Modelling Tutorials

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Tutorial Plan

1. Basics of model building

2. Group work on modelling projects

3. Group presentations of modelling problems

Basics of Model Building

- The Principle of Mass Balance
- Typical flux/motility terms
- Typical source/sink terms
- Techniques for model assembly

The Principle of Mass Balance

In 1D cartesian geometry, the net rate at which a species $n(x,t)$ accumulates satisfies

$$\frac{\partial n}{\partial t} = -\frac{\partial J}{\partial x} + S$$

where J=flux of n

and S=net rate at which n is produced

In 3D geometry, the principle states:

$$\frac{\partial n}{\partial t} = -\nabla.\boldsymbol{J} + S$$

Typical Flux Terms

- Random motion: motion down population gradients

$$J_{rm} = -\mu \frac{\partial n}{\partial x} \qquad [\boldsymbol{J}_{rm} = -\mu \nabla n]$$

- Advection with velocity $v(x,t)$

$$J_{adv} = -nv \qquad [\boldsymbol{J}_{adv} = -n\boldsymbol{v}]$$

- Chemotaxis: motion up spatial gradients of a diffusible chemical

$$J_{chemo} = \chi n \frac{\partial c}{\partial x} \quad \text{where} \quad \frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} + k(c,n)$$

$$\boldsymbol{J}_{chemo} = \chi n \nabla c \quad \text{where} \quad \frac{\partial c}{\partial t} = D\nabla^2 c + k(c,n)$$

Typical Flux Terms

- Haptotaxis: motion up spatial gradients of a non-diffusible chemical

$$J_{hapt} = \Theta n \frac{\partial a}{\partial x} \qquad \text{where} \quad \frac{\partial a}{\partial t} = f(a,n)$$

$$\boldsymbol{J}_{hapt} = \Theta n \nabla a \qquad \text{where} \quad \frac{\partial a}{\partial t} = f(a,n)$$

- Passive convection: dogs D move with velocity v and transport fleas F at the same speed:

$$J_{dog} = Dv \quad \text{and} \quad J_{fleas} = Fv \quad \Rightarrow \quad J_{fleas} = \frac{F}{D} J_{dog}$$

If dogs move by random motion and chemotaxis then

$$J_{dog} = -\mu \frac{\partial D}{\partial x} + \chi D \frac{\partial c}{\partial x} \quad \text{and} \quad J_{fleas} = -\frac{\mu F}{D}\frac{\partial D}{\partial x} + \chi F \frac{\partial c}{\partial x}$$

Typical Source/Sink Terms

- Logistic growth (source term)

$$S_{log} = rn\left(1 - \frac{n}{n_0}\right)$$

where r = population growth rate
and n_0 = carrying capacity

- Modified logistic growth (source term)

$$S_{mod,log} = rfn\left(1 - \frac{n}{n_0}\right)$$

where f = concentration of food available to population

Note: $f = 0 \Rightarrow S_{mod,log} = 0$

Typical Source/Sink Terms

- Natural decay (sink term)

$$S_{nd} = -\lambda n$$

where λ = species decay rate

- Saturable consumption (of food f by birds b, say) (sink term)

$$S_{sat} = -\frac{\lambda_0 b f}{\lambda_1 + f}$$

Note: if food is scarce ($f \to 0$) then the birds eat as much food as they can find:

$$S_{sat} \to -\frac{\lambda_0 b f}{\lambda_1} \qquad \text{as} \quad f \to 0$$

Note: if food is plentiful ($f \to \infty$) then the birds eat at a steady rate which depends on how many birds are present:

$$S_{sat} \to -\lambda_0 b \qquad \text{as} \quad f \to \infty$$

Techniques for Model Assembly

- Identify the independent and dependent variables
- For each dependent variable,
  - what factors govern its evolution?
  - what flux terms are operating?
  - what source and sink terms need to be included?
- Formulate mathematical expressions for each term that will appear in the model
- Combine the various terms using the Principle of Mass Balance

Example: Fisher's Equation

$$\frac{\partial n}{\partial t} = \mu \frac{\partial^2 n}{\partial x^2} + rn \left(1 - \frac{n}{n_0} \right)$$

Example: Fisher's Equation

$$\frac{\partial n}{\partial t} = \mu \frac{\partial^2 n}{\partial x^2} + rn\left(1 - \frac{n}{n_0}\right)$$

In this case, we have,

$$J = -\mu \frac{\partial n}{\partial x} \quad \text{and} \quad S = rn\left(1 - \frac{n}{n_0}\right)$$

i.e. random motion and logistic growth