Practice Problems - P.K. Maini

1. The variables x and y (which can take negative values), satisfy the coupled system:

$$\epsilon \frac{dx}{dt} = y - f(x), \quad \frac{dy}{dt} = -x \tag{1}$$

where $0 < \epsilon << 1$ and $f(x) = x^3 - x$.

(a) Which is the slow variable and which is the fast variable?

(b) Draw the nullclines.

(c) Sketch the phase trajectories corresponding to a relaxation oscillator, carefully indicating the fast and slow parts of the trajectory.

(d) Sketch x and y as functions of t.

2. Two dimensionless activator-inhibitor mechanisms have reaction kinetics described by

$$(i) \quad \frac{\mathrm{d}u}{\mathrm{d}t} = a - b\,u + \frac{u^2}{v}\ , \quad \frac{\mathrm{d}v}{\mathrm{d}t} = u^2 - v\ ,$$
$$(ii) \quad \frac{\mathrm{d}u}{\mathrm{d}t} = a - u + u^2\,v\ , \quad \frac{\mathrm{d}v}{\mathrm{d}t} = b - u^2\,v\ ,$$

where a and b are positive constants. Which is activator and which the inhibitor in each of (i) and (ii)? What phenomena are indicated by the nonlinear terms? Sketch the null clines. Is it possible to have multiple positive steady states with these kinetics? Discuss the stability of any non-zero steady states in (i). What can you say about the number of non-zero steady states if substrate inhibition is included in (i), that is u^2/v is replaced by $u^2/[v(1 + k\,u^2)]$?

3. Suppose fishing is regulated in a zone Hkm from a country's shore (taken to be a straight line), but outside this zone over-fishing is so excessive that the population is effectively zero. Assume that the fish reproduce logistically, disperse by diffusion and within the zone are harvested with an effort E. Justify the following model for the fish population $u(x,t)$:

$$u_t = ru\left(1 - \frac{u}{K}\right) - EU + Du_{xx},$$

with boundary conditions

$$u = 0 \text{ on } x = H,\ u_x = 0 \text{ on } x = 0,$$

where r, K, $E(< r)$ and D are positive constants.

If the fish stock is not to collapse, show that the fishing zone H must be greater than $\pi/2[D/(r-E)]^{\frac{1}{2}}$km. Briefly discuss any ecological implications.

4. Consider the reaction-diffusion system

$$\frac{\partial u}{\partial t} = f(u,v) + D_1 u_{xx},$$

$$\frac{\partial v}{\partial t} = g(u,v) + D_2 v_{xx},$$

where f and g describe the reaction kinetics, D_1 and D_2 are positive constants. Derive the following necessary conditions for diffusive-driven instability:

(i) $f_u + g_v < 0$

(ii) $f_u g_v - f_v g_u > 0$

(iii) $D_2 f_u + D_1 g_v > 0$

(iv) $D_2 f_u + D_1 g_v > 2[D_1 D_2 (f_u g_v - f_v g_u)]^{1/2}$

(v) $k_1 < \frac{n^2\pi^2}{L^2} < k_2$

where the partial derivatives are evaluated at the uniform steady state, n is a positive integer, and k_1, k_2 are constants you should find.

Show that

(A) Conditions (i) and (iii) imply that D_1 does not equal D_2.

(B) Conditions (i), (ii) and (iii) impose on the Jacobian matrix at most two different sign structures.

(C) Under these conditions bifurcation to solutions oscillating in time as well as space (Hopf bifurcation) cannot occur.

By referring to condition (v), describe how the form of the patterns changes as L increases.

5. Consider the Gierer-Meinhardt reaction-diffusion system in one dimension:

$$\frac{\partial A}{\partial t} = \frac{\rho A^2}{(1 + KA^2)H} - \mu A + D_A A_{xx},$$

$$\frac{\partial H}{\partial t} = \rho' A^2 - \nu H + D_H H_{xx},$$

where A and H are the reactants, ρ, K, μ, ν, ρ', D_A and D_H are positive constants.

(a) Draw a phase portrait of this scheme in the absence of diffusion and show that diffusive driven instability may be possible if the nullclines intersect in a certain way.

(b) Write down the conditions for diffusive driven instability.

[In (a) and (b) consider only non-zero steady states.]

6. The amoebae of the slime mold *Dictyostelium discoideum,* with density $n(x,t)$, secrete a chemical attractant, cyclic-AMP, and spatial aggregations of amoebae start to form. One of the models for this process gives rise to the system of equations which, in their one-dimensional form, are

$$n_t = D_n n_{xx} - \chi(n a_x)_x,$$

$$a_t = hn - ka + D_a a_{xx},$$

where a is the attractant concentration and h, k, χ and the diffusion coefficients D_n and D_a are all positive constants.

Nondimensionalize the system to obtain

$$n_t = D_n n_{xx} - \chi(n a_x)_x,$$

$$a_t = n - a + D_a a_{xx},$$

where the variables and parameters are now nondimensional. Then consider (i) a finite domain with zero flux boundary conditions and (ii) an infinite domain. Examine the linear stability about the steady state (which introduces a further parameter here), derive the dispersion relation and discuss the role of the various parameter groupings. Hence obtain the conditions on the parameters and domain size for the mechanism to initiate spatially heterogeneous solutions.

Experimentally, the chemotactic parameter χ increases during the life cycle of the slime mold. Using χ as the bifurcation parameter, determine the critical wave length when the system bifurcates to spatially structured solutions in the finite domain. Also examine the bifurcating instability as the domain length increases. Briefly describe the physical process operating and explain intuitively how spatial aggregation takes place.

7. A population $u(\mathbf{x}, t)$ has diffusion coefficient $D(u)$ and production rate $g(u)$ per unit volume, where $D(u) > 0$, $\forall u$. Show that, as a result of Fick's Law, u satisfies the reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \nabla.(D(u)\nabla u) + g(u).$$

Now suppose that $g(0) = g(1) = 0$, $g(u) > 0$, $\forall u \in (0, 1)$, $g'(0) > 0$, $g'(1) < 0$, and g and D are continuously differentiable. Show that, for the one-dimensional case, if a travelling wave $u = \phi(\xi)$, $\xi = x - ct$, exists from $u = 1$ to $u = 0$, then

$$c = \frac{\int_0^1 g(w)D(w)dw}{\int_{-\infty}^{\infty} D(\phi(s)) \left[\frac{d\phi(s)}{ds}\right]^2 ds},$$

and hence that $c > 0$. (Hint: Convert to travelling wave coordinates, then multiply by $D\phi'$ and integrate.)

Assuming that such a travelling wave solution is possible, find the lower limit on the wave speed. Sketch $u(x)$ for the travelling wave solution, with the direction of motion clearly marked.

8. In the reaction sequence

$$X + Y \overset{k_1}{\longrightarrow} P, \quad A + X \overset{k_2}{\longrightarrow} 2X, \quad 2X \overset{k_3}{\longrightarrow} P + A,$$

the reactants can diffuse in a one-dimensional space with the same diffusion coefficient D. Write down the reaction-diffusion system for X and Y using the law of mass action and with the concentration A constant.

Using the non-dimensionalisation

$$u = \frac{2k_3[X]}{k_2[A]}, \; v = \frac{k_1[Y]}{k_2[A]r}, \; x^* = \left(\frac{k_2[A]}{D}\right)^{1/2} x, \; t^* = k_2[A]t \text{ and } b = \frac{k_1}{2k_3},$$

where r is a positive parameter, show that the reaction-diffusion system becomes

$$\frac{\partial u}{\partial t^*} = u(1 - u - rv) + \frac{\partial^2 u}{\partial x^{*2}}, \; \frac{\partial v}{\partial t^*} = -buv + \frac{\partial^2 v}{\partial x^{*2}}.$$

This system exhibits travelling wave front solutions with $u(-\infty, t^*) = v(\infty, t^*) = 1$ and $u(\infty, t^*) = v(-\infty, t^*) = 0$. Show that in the symmetric situation $u = 1 - v$, the travelling wave fronts are solutions of a Fisher equation, if $b = 1 - r$. Hence, show that the travelling front wave speed c satisfies the inequality $c \geq 2\sqrt{1-r}$, if $0 \leq r \leq 1$.

9. A model for travelling bands of bacteria in one dimension takes the form

$$\frac{\partial b}{\partial t} = \frac{\partial}{\partial x}\left[D\frac{\partial b}{\partial x} - \frac{Xb}{a}\frac{\partial a}{\partial x}\right],$$

$$\frac{\partial a}{\partial t} = -kb,$$

where $b(x,t)$ and $a(x,t)$ are bacteria density and chemo-attractant concentration, respectively, at a position x and time t, and D, X and k are positive parameters which should be assumed constant. Briefly explain the biological meaning of each term.

Derive equations satisfied by travelling wave solutions of the form $a(z)$, $b(z)$, $z = x - ct$ (c constant) joining $(a,b) = (0,0)$ at $z \to -\infty$ to $(a,b) = (1,0)$ at $z \to \infty$, and find the relation between $b(z)$ and $a(z)$.

Hence, for the case $\frac{X}{D} = 2$, find b and a explicitly in terms of z, showing carefully that your solution satisfies the boundary conditions. For the case $a(0) = \frac{1}{2}$, show that $b(z)$ is symmetric about the axis $z = 0$. Sketch a and b as functions of z and briefly describe what is happening biologically.

10. A rabies model which includes a logistic growth for the susceptibles S and diffusive dispersal for the infectives is

$$\frac{\partial S}{\partial t} = -rIS + BS(1 - \frac{S}{S_0}),$$

$$\frac{\partial I}{\partial t} = rIS - aI + D\frac{\partial^2 I}{\partial x^2},$$

where r, B, a, D and S_0 are positive constant parameters. Nondimensionalise the system to give

$$u_t = u_{xx} + uv - \lambda u,$$

$$v_t = -uv + bv(1-v),$$

where u relates to I and v to S, and x and t now denote nondimensionalised spatial and temporal coordinates. Look for travelling wave solutions with $u > 0$ and $v > 0$ and hence show, by linearising far ahead of a wave front where $v \to 1$ and $u \to 0$, *i.e.* far ahead where the population is still fully susceptible and the infection has not yet arrived, that a wave may exist if $\lambda < 1$ and, if so, the minimum wave speed is $2(1-\lambda)^{\frac{1}{2}}$.