Motivation for the Laplace Transform

• CT Fourier transform enables us to do a lot of things, e.g.
  — Solving LCCDE’s
  — Analyzing frequency response of LTI systems
  — Sampling
  — Modulation
  :

• Why do we need yet another transform?

• Fourier transform cannot handle unstable signals/systems. i.e.
  \[ \int_{-\infty}^{\infty} |x(t)| \, dt = \infty \]

Because the eigenfunction \( e^{j\omega t} \) in Fourier transform has unit amplitude \( |e^{j\omega t}| = 1 \). For example:
Motivation for the Laplace Transform (continued)

\[ x(t) \rightarrow h(t) \rightarrow y(t) \]

And \( h(t) = e^t u(t) \)
— an unstable causal system

\[ H(j\omega) = ? \]

Consequently, we cannot analyze this system using the easy method of

\[ Y(j\omega) = H(j\omega)X(j\omega). \]

Do we have to go back to \( y(t) = h(t) \ast x(t) \)?

Motivation for the Laplace Transform (continued)

- In many applications, we do need to deal with unstable systems, e.g.
  - Stabilizing an inverted pendulum
  - Stabilizing an airplane or space shuttle
  
  \[ \vdots \]
  - Instability is desired in some applications, e.g. oscillators and lasers

- How do we analyze unstable signals/systems?
Recall from Lecture #5, eigenfunction property of LTI systems:

\[ e^{st} \rightarrow h(t) \rightarrow H(s)e^{st} \]

\[ H(s) = \int_{-\infty}^{\infty} h(t)e^{-st} dt \]  (assuming this converges)

- \( e^{st} \) is an eigenfunction of any LTI systems,
- \( s = \sigma + j\omega \) can be complex in general
The (Bilateral) Laplace Transform

(Can be viewed as an extension of the Fourier Transform)

\[ x(t) \leftrightarrow X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}dt = \mathcal{L}\{x(t)\} \]

\( s = \sigma + j\omega \) is a complex variable — Now we explore the full range of \( s \)

First comments:

1. \[ X(\sigma + j\omega) = \int_{-\infty}^{+\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}dt = \mathfrak{R}\{x(t)e^{-\sigma t}\} \]

2. A critical issue in dealing with Laplace transform is convergence:
   — \( X(s) \) generally exists only for some values of \( s \), located in what is called the region of convergence (ROC)
   \[ \text{ROC} \subseteq \{s = \sigma + j\omega \text{ so that } \int_{-\infty}^{+\infty} |x(t)e^{-\sigma t}|dt < \infty \} \]
   Depends only on \( \sigma \) not on \( \omega \)

3. If \( s = j\omega \) is in the ROC (i.e. \( \sigma = 0 \)), then
   \[ \mathcal{L}\{x(t)\}|_{s = j\omega} = X(s)|_{s = j\omega} = \mathfrak{R}\{x(t)\} \]

Example #1:

\[ x_1(t) = e^{-at}u(t) \]

\( a \) — an arbitrary real or complex number

Unstable, no FT but LT exists

\[ X_1(s) = \int_{-\infty}^{+\infty} e^{-at}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-(s+a)t}dt \]

\[ = -\frac{1}{s + a}e^{-(s+a)t}\bigg|_{0}^{\infty} = -\frac{1}{s + a} [ e^{-(s+a)\infty} - 1 ] \]

This converges only if \( \text{Re}(s+a) > 0 \), i.e. \( \text{Re}(s) > -\text{Re}(a) \)

\[ \downarrow \quad X_1(s) = \frac{1}{s + a} , \quad \frac{\text{Re}(s) > -\text{Re}(a)}{\text{ROC}} \]
Example #2:

\[ x_2(t) = -e^{-at}u(-t) \]

\[ X_2(s) = - \int_{-\infty}^{0} e^{-at}u(-t)e^{-st} \, dt \]

\[ = - \int_{-\infty}^{0} e^{-(s+a)t} \, dt \]

\[ = \frac{1}{s + a} \left[ 1 - e^{-(s+a)t} \right] \bigg|_{-\infty}^{0} = \frac{1}{s + a} \left[ 1 - e^{-2(s+a)} \right] \]

This converges only if \( Re(s+a) < 0 \), i.e. \( Re(s) < -Re(a) \)

\[ \Downarrow \]

\[ X_2(s) = \frac{1}{s + a} , \quad Re(s) < -Re(a) \quad \text{Same as } X_1(s), \text{ but a different } ROC \]

**Key Point** (and a key difference with \( FT \)): Need both \( X(s) \) and ROC to uniquely determine \( x(t) \). No such an issue of ROC for \( FT \).

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**Graphical Visualization of the ROC**

**Example #1**

\[ x_1(t) = e^{-at}u(t) \quad \text{— right-sided signal,} \]

\[ X_1(s) = \frac{1}{s + a} , \quad Re(s) > -Re(a) \]

**Example #2**

\[ x_2(t) = -e^{-at}u(-t) \quad \text{— left-sided signal} \]

\[ X_2(s) = \frac{1}{s + a} , \quad Re(s) < -Re(a) \]
Rational Transforms

• Many (but by no means all) Laplace transforms of interest to us are rational functions of \( s \) (e.g. Examples #1 & #2, in general, LTIs described by LCCDEs), i.e.

\[
X(s) = \frac{N(s)}{D(s)}, \quad N(s), D(s) — \text{polynomials in } s
\]

• Roots of \( N(s) = \text{zeros of } X(s) \)

• Roots of \( D(s) = \text{poles of } X(s) \)

• Any \( x(t) \) consisting of a linear combination of complex exponentials for \( t > 0 \) and for \( t < 0 \) (i.e. as in Example #1 and #2) has a rational Laplace transform.

Example #3 \( x(t) = 3e^{2t}u(t) - 2e^{-t}u(t) \)

\[
X(s) = \int_{0}^{\infty} [3e^{2t} - 2e^{-t}]e^{-st} \, dt
\]

\[
= 3 \int_{0}^{\infty} e^{-(s-2)t} \, dt - 2 \int_{0}^{\infty} e^{-(s+1)t} \, dt
\]

\[
= \frac{3}{s-2} - \frac{2}{s+1} = \frac{s+7}{(s-2)(s+1)} \quad \text{Re}(s) > 2
\]

\( \Rightarrow \) two poles \( (p_1 = 2, p_2 = -1) \) & one zero \( (z_1 = -7) \)

\[\text{Notation:} \quad \times — \text{pole} \quad \circ — \text{zero} \]

Q: Does \( x(t) \) have FT?
Laplace Transforms and ROC’s

- First of all, some signals do not have Laplace transforms

(a) \( x(t) = Ce^{-t} \) for all \( t \) since \( \int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| \, dt = \infty \) for all \( \sigma \)

(b) \( x(t) = e^{j\omega t} \) for all \( t \)

\[
\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| \, dt = \int_{-\infty}^{\infty} e^{-\sigma t} \, dt = \infty \text{ for all } \sigma
\]

\( X(s) \) is defined only in ROC, by definition, \( X(s) \neq \infty \Rightarrow \) No \( \delta(s) \) is allowed, different from FT

Properties of the ROC

Secondly, the ROC can take on only a small number of different forms

1) The ROC consists of a collection of lines parallel to the \( j\omega \)-axis in the \( s \)-plane (i.e. the ROC only depends on \( \sigma \)).

Why?

\[
\int_{-\infty}^{\infty} |x(t)e^{-st}| \, dt = \int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| \, dt < \infty \text{ depends only on } \sigma = \text{Re}(s)
\]

2) If \( X(s) \) is rational, then the ROC does not contain any poles.

Why?

Poles are places where \( D(s) = 0 \)

\[
\Rightarrow X(s) = \frac{N(s)}{D(s)} = \infty \text{ Not convergent.}
\]
More Properties

3) If \( x(t) \) is of finite duration and is absolutely integrable, then the ROC is the entire \( s \)-plane.

\[
X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt = \int_{T_1}^{T_2} x(t)e^{-st} dt
\]

< \infty if \( \int_{T_1}^{T_2} |x(t)| dt < \infty \)

ROC Properties that Depend on which Side you are On

4) If \( x(t) \) is right-sided (i.e. if it is zero before some time), and if \( Re(s) = \sigma_0 \) is in the ROC, then all values of \( s \) on the right side of the vertical line \( Re(s) = \sigma_0 \) are also in the ROC.
ROC Properties that Depend on which Side you are On

5) If \( x(t) \) is left-sided (i.e. if it is zero after some time), and if \( Re(s) = \sigma_o \) is in the ROC, then all values of \( s \) on the left side of the vertical line \( Re(s) = \sigma_o \) are also in the ROC.

Still More ROC Properties

6) If \( x(t) \) is two-sided and if the line \( Re(s) = \sigma_o \) is in the ROC, then the ROC consists of a strip in the \( s \)-plane that includes the line \( Re(s) = \sigma_o \).
Example: 

\[ x(t) = e^{-b|t|} \]

Example (continued):

\[ x(t) = e^{bt}u(-t) + e^{-bt}u(t) \]

\[ \downarrow \quad \uparrow \]

\[ -\frac{1}{s-b}, \text{Re}\{s\} < b \quad \frac{1}{s+b}, \text{Re}\{s\} > -b \]

Overlap only if \( b > 0 \)  \( \Rightarrow \) \( X(s) = \frac{2b}{s^2 - b^2} \), with ROC

What if \( b < 0 \)?  \( \Rightarrow \) No overlap  \( \Rightarrow \) No Laplace Transform
Properties, Properties

7) If \(X(s)\) is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of \(X(s)\) are contained in the ROC.

8) Suppose \(X(s)\) is rational, then
   (a) If \(x(t)\) is right-sided, the ROC is to the right of the rightmost pole.
   (b) If \(x(t)\) is left-sided, the ROC is to the left of the leftmost pole.

   \[\begin{align*}
   &\text{right-sided} & \text{left-sided} \\
   &\text{(a)} & \text{(b)}
   \end{align*}\]

9) If ROC of \(X(s)\) includes the \(j\omega\)-axis, then \(FT\) of \(x(t)\) exists.

Example: \[X(s) = \frac{(s + 3)}{(s + 1)(s - 2)}\]

Two poles \(\Rightarrow\) three possible ROC’s

\[\begin{align*}
\text{ROC: III} & \quad \text{ROC: I} \\
\text{ROC: II} & \quad \text{ROC: II}
\end{align*}\]
Next lecture covers:
O&W pp. 670-698