# MATH20902: Discrete Maths, Solutions to Problem Set 9

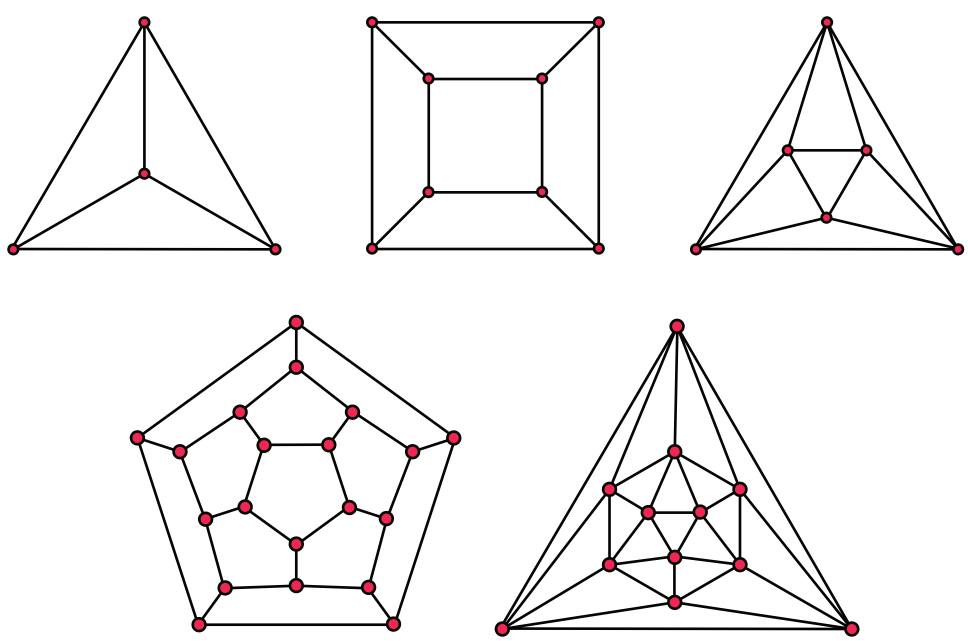
**(1)** (Euler's theorem and graphs from regular polyhedra)**.**
One can actually check Euler's theorem without drawing all the graphs. The polyhedra in question—tetrahedrons, cubes, octahedrons, dodecahedrons and icosahedrons—are all characterized by having a certain number of identical faces, each of which is a regular polygon (all edges the same length, all internal angles equal). Further, a certain fixed number of these identical faces meet at each vertex: the properties of these so-called *Platonic solids* are summarized in the table below:

| *Name* | *Face* | *Faces meeting at a vertex* | *Vertices* $n$ | *Edges* $m$ | *Faces* $f$ |
|---|---|---|---|---|---|
| Tetrahedron | Triangle | 3 | 4 | 6 | 4 |
| Cube | Square | 3 | 8 | 12 | 6 |
| Octahedron | Triangle | 4 | 6 | 12 | 8 |
| Dodecahedron | Pentagon | 3 | 20 | 30 | 12 |
| Icosahedron | Triangle | 5 | 12 | 30 | 20 |

It's easy to see that Euler's theorem,

$$n - m + f = 2,$$

is satisfied for all five. As is usual with planar graphs, there are many ways to draw the planar diagrams, but the figure below illustrates one suitable collection.

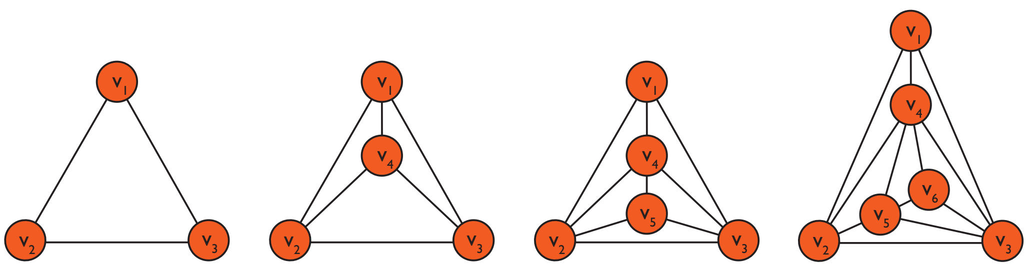


**(2)** (After Jungnickel's exercise 1.5.13).
We know for that for a planar graph with $n$ vertices and $m$ edges the bound $m \leq 3n - 6$ applies. Now, the complete graph on $n$ vertices has $n(n-1)/2$ edges, so, at the very least, one needs to remove

$$\frac{n(n-1)}{2} - (3n - 6) = \frac{n^2 - 7n + 12}{2} \tag{2.1}$$

edges to get a planar graph.

This logic provides a lower bound on the number of edges we need to remove to make $K_n$ planar, but one can prove more. The bound above is sharp in the sense that it is possible to construct a planar graph on $n$ vertices with exactly $m = 3n - 6$ edges. The figure below illustrates the main idea, which allows one to construct a sequence of graphs recursively.

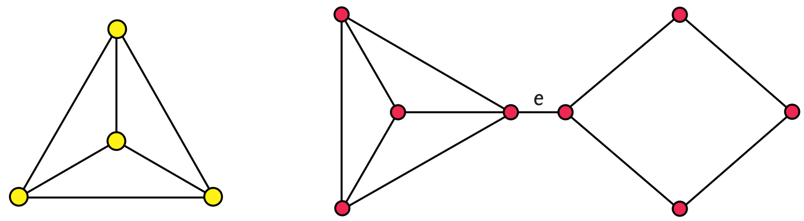



The base case is $n = 3$, when the bound (2.1) says that $m \leq 3$ and the upper limit is attained in $K_3$, the complete graph on three vertices. If we number the vertices as shown in the figure, it's clear that—by adding a vertex that is adjacent to vertices $v_1$, $v_2$ and $v_3$—we can make a planar graph on $n = 4$ vertices which also has the maximal number of edges. Further, this graph has a planar diagram with a triangular face bounded by the three-cycle $(v_2, v_3, v_4, v_2)$. One can continue in this way, adding a new vertex $v_{k+1}$ "in the middle" of the face bounded by the three-cycle

$$(v_{k-2},\, v_{k-1},\, v_k,\, v_{k-2})$$

and, as each step of the construction adds one vertex and three edges, we will obtain a sequence of triangulated planar graphs with $n$ vertices and $m = 3n - 6$ edges.

**(3)** (Edges and bridges)**.** (a) There are no bridges in the graph at left (yellow vertices) in the figure below and just one, labelled $e$, in the graph at right (red vertices).




(b) This is very similar to one of the exercises about trees. Suppose that in a connected graph $G$ with vertex set $V$ and edge set $E$, the edge $e = (a, b) \in E$ is not a bridge. Then if we delete $e$ to form a new graph $H = G\backslash e$ this new graph is also connected. In particular, there is a path

$$(a = v_0,\, v_1,\, \ldots,\, v_k = b)$$

connecting the vertices $a$ and $b$, where all the edges $(v_j, v_{j+1})$ for $0 \leq j < k$ are present in both $H$ and $G$. But then a cycle

$$(a = v_0,\, v_1,\, \ldots,\, v_k,\, v_0 = a)$$

is present in $G$, where the final edge, the one connecting $v_k = b$ and $v_0 = a$, is $e$.

**(4)** (After Jungnickel's exercise 1.5.14)**.**
When approaching a problem like this, a good way to start is to write down everything you know. In this case, as $G$ is a planar graph with $n \geq 3$ vertices, we have the following bound on the number of edges:

$$m \leq 3n - 6. \tag{4.1}$$

Of course, we also have a potentially sharper bound that involves the girth of $G$, but this problem says nothing about girth. In addition to the inequality above, we have one other fact, which is true of all graphs. If $G$ has vertex set $V$, we can use the Handshaking Lemma to write

$$\sum_{v \in V} \deg(v) = 2m \qquad \text{or} \qquad \frac{1}{2} \sum_{v \in V} \deg(v) = m. \tag{4.2}$$

As problem asks about $n_d$, the number of vertices whose degree is less than or equal to $d$, it will prove convenient to have a notation for the number of vertices whose degree is exactly $j$. Let's define $\ell_j$ to be

$$\ell_j = |\{v \in V \,|\, \deg(v) = j\}|\,.$$

It's now easy to write down formulae for $n_d$ and for the total number of vertices, $n$:

$$n_d = \sum_{j=0}^{d} \ell_j \qquad \text{and} \qquad n = \sum_{j=0}^{n-1} \ell_j.$$

We can also use the $\ell_j$ to rewrite the relation in (4.2), which becomes

$$m = \frac{1}{2} \sum_{j=0}^{n-1} j\, \ell_j.$$

Putting this result together with (4.1) yields

$$3n - 6 \;\geq\; \frac{1}{2} \sum_{j=0}^{n-1} j\, \ell_j. \qquad \text{or} \qquad \sum_{j=0}^{n-1} j\, \ell_j \;\leq\; 6n - 12, \tag{4.3}$$

which is beginning to look a bit like the thing we are trying to prove. To complete the argument we need to look more closely at the sum in the expressions above. The first step is to break the sum into two pieces: one involving the vertices of degree no more than $d$ and the other for those of higher degree:

$$\sum_{j=0}^{n-1} j\, \ell_j = \left( \sum_{j=0}^{d} j\, \ell_j \right) + \left( \sum_{j=(d+1)}^{n-1} j\, \ell_j \right). \tag{4.4}$$

Now we can obtain a lower bound on these sums by replacing the factor of $j$ in each with its smallest value:

$$\begin{aligned} \left( \sum_{j=0}^{d} j\, \ell_j \right) + \left( \sum_{j=(d+1)}^{n-1} j\, \ell_j \right) \;&\geq\; \left( \sum_{j=0}^{d} 0 \times \ell_j \right) + \left( \sum_{j=(d+1)}^{n-1} (d+1) \times \ell_j \right) \\ &\geq\; (d+1) \sum_{j=(d+1)}^{n-1} \ell_j \\ &\geq\; (d+1)(n - n_d), \end{aligned} \tag{4.5}$$

where the last line follows because the sum $\sum_{j=(d+1)}^{n-1} \ell_j$ counts those vertices whose degree exceeds $d$ and, as there are $n$ vertices in total, there are exactly $(n - n_d)$ such high-degree vertices.

Finally, putting Eqns (4.3)–(4.5) together, we have

$$(d+1)(n - n_d) \;\leq\; \left( \sum_{j=0}^{d} j\, \ell_j \right) + \left( \sum_{j=(d+1)}^{n-1} j\, \ell_j \right) \;\leq\; 6n - 12$$

or, tidying things up,

$$\begin{aligned}
(d+1)(n-n_d) &\leq 6n-12 \\
nd+n-(d+1)n_d &\leq 6n-12 \\
-(d+1)n_d &\leq 5n-dn-12 \\
-(d+1)n_d &\leq (5-d)n-12 \\
(d+1)n_d &\geq (d-5)n+12 \\
n_d &\geq \frac{(d-5)n+12}{d+1}
\end{aligned}$$

just as advertised.

**(5)** (Direct proofs that $K_5$ and $K_{3,3}$ aren't planar)**.** The main idea needed for this problem is that if a vertex $v$ lies in the interior of some Jordan curve $C$, while a second vertex $u$ lies in the exterior of $C$, then any curve that represents an edge connecting $u$ to $v$ must intersect $C$.

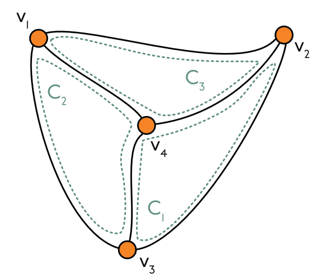



Figure 1: *The three cycles $C_1$, $C_2$ and $C_3$ used in the proof that $K_5$ can't have a planar diagram.*

(a) The various cycles mentioned in the problem are illustrated in Figure 1: they're defined so that the vertex $v_j$ lies in the exterior of $C_j$. Thus, for example, $v_1 \in \text{ext}(C_1)$ and $v_3 \in \text{ext}(C_3)$.

Consider the problem of adding the vertex $v_5$ to the diagram in Figure 1 in such a way as to get a planar diagram for $K_5$. The point representing $v_5$ must lie in the exterior of $C_1$, as otherwise the curve representing the egde $(v_1, v_5)$ would have to cross $C_1$. Similar arguments show $v_5 \in \text{ext}(C_2)$ and $v_5 \in \text{ext}(C_3)$, so

$$v_5 \in \bigcap_{j=1}^{3} \text{ext}(C_j), \tag{5.1}$$

which is the result we sought.

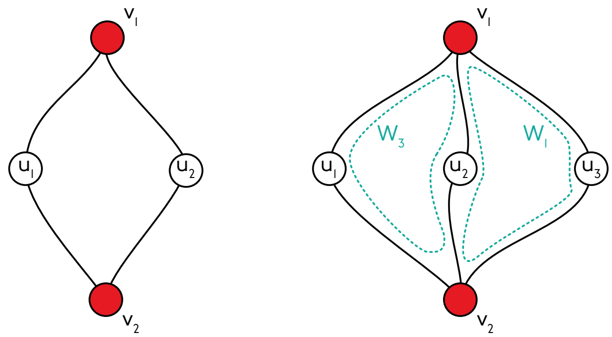



Figure 2: *At left, the cycle $(u_1, v_1, u_2, v_2, u_1)$ and at right, a planar diagram for* $K_{3,3} \backslash \{v_3\}$.

(b) Now consider the Jordan curve that represents the edges $\{(v_1, v_2), (v_2, v_3), (v_3, v_1)\}$. On the one hand, $v_4 \in \text{int}(C)$ and, on the other hand, Eqn. (5.1) implies that $v_5 \in \text{ext}(C)$. But then any curve that represents the edge $(v_4, v_5)$ must cross $C$, which means that it's impossible to find a planar diagram for $K_5$.

(c) One can prove that $K_{3,3}$ is nonplanar in a similar way. To be concrete, say that the vertex set of $K_{3,3}$ is

$$V = \{u_1, u_2, u_3, v_1, v_2, v_3\}$$

and that the edge set includes all possible edges of the form $(u_j, v_k)$. The left panel of Figure 2 then shows a planar diagram for the cycle that includes all such edges running between vertices $u_1, v_1$, $u_2$ and $v_2$: one can also think of it as a planar diagram for $K_{2,2}$. The right panel of Figure 2 shows the result of adding one more vertex, $u_3$, and it's clear that, up to renumbering of the vertices, any planar diagram for $K_{3,3} \backslash \{v_3\}$—which is isomorphic to $K_{3,2}$—must look similar.

The right panel of Figure 2 also illustrates two cycles, $W_1$ and $W_3$, with the properties that $u_j \in \text{ext}(W_j)$. Now think about adding $v_3$ to the diagram: the point representing $v_3$ must be in the exterior of $W_1$ (so that we can draw the edge $(u_1, v_3)$ without crossing any other edges) and, for similar reasons involving $u_3$ and the edge $(u_3, v_3)$, the point representing $v_3$ must also lie in the exterior of $W_3$. Thus the analogue of Eqn. (5.1) is

$$v_3 \in \text{ext}(W_1) \bigcap \text{ext}(W_3).$$

Finally, consider the Jordan curve $W$ that represents the cycle $(u_1, v_1, u_3, v_2, u_1)$. The reasoning in the preceeding paragraph says $v_3 \in \text{ext}(W)$, but $u_2 \in \text{int}(W)$, so any curve representing the edge $(u_2, v_3)$ must cross $W$, which implies that $K_{3,3}$ can't have a planar diagram.