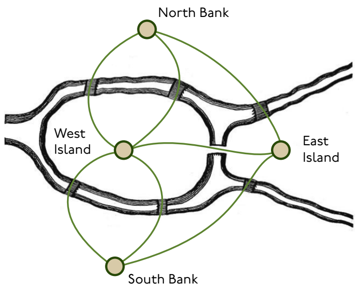
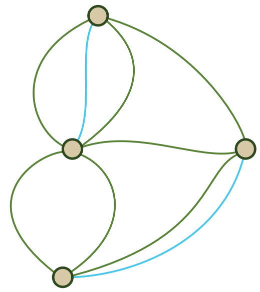
MATH20902: Discrete Maths, Solutions to Problem Set 7

(1) (after Marcus's F5).

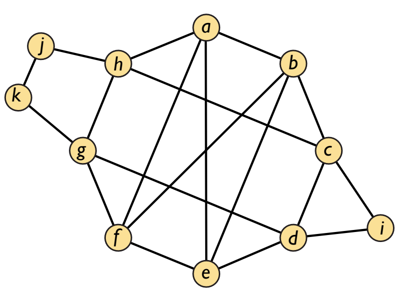


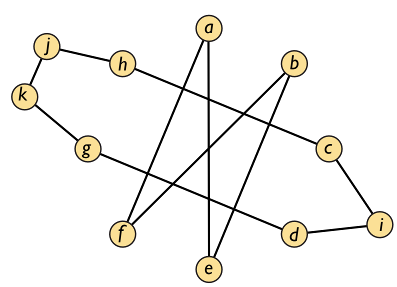


The figure at left above includes a sketch-map of Königsberg from Euler's original paper that shows the city's two islands and seven bridges, along with a multigraph that captures the city's topology and contains no Eulerian tour. All four vertices in this multigraph have odd degree, while in an Eulerian multigraph every vertex must have even degree.

Define δ to be the minimal number of edges we can add to the multigraph to make it Eulerian. Adding an edge to a multigraph increases the degree of each its endpoints by one and so we need to add at least two edges to alter the degrees of all four of the vertices: this establishes that $\delta \geq 2$. On the other hand, the figure at right shows a pair of edges whose addition causes every vertex to have even degree. This explicit example implies that δ, which is defined as a minimum over all possible additions of edges, satisfies $\delta \leq 2$. Finally, these two bounds taken together imply that $\delta = 2$.

(2) (after Marcus's F9).





(a) The orignal graph is at left above, while the one with the cycle $\mathcal{C}_1 = (a, b, c, d, e, f, g, h, a)$ removed is at right.

(b) There are two components, one containing the cycle $\mathcal{C}_2 = (a, e, b, f, a)$ and another containing the cycle $\mathcal{C}_3 = (c, i, d, g, k, j, h, c)$.

(c) $\mathcal{C}_1$ and $\mathcal{C}_2$ both include the vertex a, so we can merge them to get the closed trail

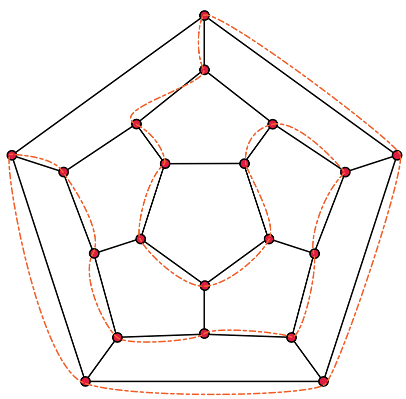
$$(a, b, c, d, e, f, g, h, a, e, b, f, a).$$

Then, as both the trail and $\mathcal{C}_3$ include the vertex c, we can merge them to get

$$(a, b, c, i, d, g, k, j, h, c, d, e, f, g, h, a, e, b, f, a),$$

which is an Eulerian tour for the original graph. There are, of course, many other ways to merge the cycles and hence many other Eulerian tours.

(3). It's easy enough to construct a Hamiltonian tour by trial-and-error: one possibility is illustrated below.



(4). This answer depends on solutions to problems from earlier in the term.

(a) A graph G on n vertices differs from its closure $[G]$ if and only if it contains at least one pair of non-adjacent vertices u and v with $\deg(v)+\deg(v) \geq n$. Now, the cube graph I_d has 2^d vertices, each of which has degree d and so $I_d = [I_d]$ provided

$$d + d < 2^d \qquad \text{or} \qquad 2d < 2^d \qquad \text{or} \qquad d \geq 3.$$

(b) The cube graph I_d is always connected, and so it is Eulerian whenever the degrees of its vertices are all even. This happens when d itself is an even number.

(c) One of the problems in Problem Set 2 asked you to show that I_d contains a subgraph isomorphic to the cycle graph C_{2^d} and the solutions provide a constructive procedure to produce such a cycle. Given that I_d has exactly 2^d vertices, the same construction yields a Hamiltonian cycle: it's just that we didn't know to call it that earlier in the term. That is, I_d is Hamiltonian for $d \geq 2$.

(5) (after Marcus's F8)**.**
The problem asks us to invent an alternative proof of the following lemma, which I proved in lecture by providing a constructive algorithm.

Lemma (Vertices of even degree and cycles)**.** *If $G(V, E)$ is a multigraph with a nonempty edge set $E \neq \emptyset$ and the property that $\deg(v)$ is an even number for all $v \in V$, then G contains a cycle.*

Proof. Assume for contradiction that $G(V, E)$ does not contain any cycles. Then it does not contain any parallel edges either, as if the edge list had two copies of some edge (u, v), the graph would contain a cycle of length two: (u, v, u). This implies that G is an ordinary, undirected graph.

As $E \neq \emptyset$, G must have at least one non-trivial connected component (non-trivial in the sense that it contains more than one vertex). Choose such a component and call it $G_1(V_1, E_1)$. It is a connected, acyclic graph—a tree. It then follows from results in the lecture titled Tree and Forests that $G_1(V_1, E_1)$ contains at least two leaves—vertices of degree one. But this provides a contradiction: all the vertices in G are supposed to have even degree. □

Yehao Liu, who did the course in 2019, came up to me after the lecture about Eulerian tours and suggested this proof, which he'd invented on the spot!

(6) (after Jungnickel's Exercise 1.3.3)**.**
Number the vertices in such a way that the $2k$ vertices of odd degree are $v_1, v_2, \ldots, v_{2k}$ and make a new multigraph, H, by adding the k edges

$$(v_1, v_2),\ (v_3, v_4),\ \ldots\ (v_{2j-1}, v_{2j})\ \ldots\ (v_{2k-1}, v_{2k})$$

with $1 \leq j \leq k$ to G. Note that as we permit H to be a multigraph (that is, have parallel edges), it doesn't matter whether some of the pairs listed above are already adjacent. Adding all these edges increases the degrees of all the odd-degree vertices by one, making them even. Thus H is Eulerian (it's connected because G is), so it has an Eulerian tour that includes all the original edges from G as well as the k new edges that made H. Now delete those k edges from the tour: as the edges we added are pairwise disjoint (that is, none of the new edges share vertices) what remains is a collection of k disjoint trails that include all the edges of G.

(7). Note that if v is a vertex in a graph $G(V, E)$ on n vertices, then $\deg(v) \leq (n-1)$. Further, if $\deg(v) = (n-1)$, then v is connected to every other vertex in the graph.

Now recall the closure construction: it adds an edge between two vertices if they aren't already adjacent and

$$\deg u + \deg v \geq n. \tag{7.1}$$

Finally, consider some vertex u with $\deg(u) < 2$. This means either $\deg(u) = 0$ or $\deg(u) = 1$: take the former case, $\deg(u) = 0$, first. The closure construction can never add an edge incident on u because, for all $v \neq u \in V$, we have

$$\deg u + \deg v \ = \ 0 + \deg v \ \leq \ n - 1,$$

so the sum of their degrees can never be big enough to require a new edge connecting u and v. On the other hand, if $\deg u = 1$ it's possible that there is some vertex $v \in V$ such that (7.1) is satisfied, but any such v would need to have $\deg v = (n-1)$, so it would already be connected to every other vertex in the graph, including u.

(8) (after Marcus's G42)**.**
We are considering graphs $G(V, E)$ with degree sequence $(2, 3, 3, 4, 4, 5, 5)$ and have partitioned the vertex set into two disjoint pieces, $\mathcal{L}$ (for "low") and $\mathcal{H}$ (for "high") defined as follows

$$\mathcal{L} = \{u \in V \mid \deg(u) \leq 3\} \quad \text{and} \quad \mathcal{H} = \{v \in V \mid \deg(v) \geq 4\}.$$

(a) Consider a vertex $u \in \mathcal{L}$. It has $|A_u| = \deg(u) \leq 3$ neighbours and so, as there are four vertices in $\mathcal{H}$, there must be at least $|\mathcal{H}| - \deg(u) = 4 - \deg(u) > 0$ vertices in $\mathcal{H}$ that are not adjacent to u.

(b) Consider first a vertex u of degree 3 in G and choose a vertex $v \in \mathcal{H}$ that is not adjacent to u: we know that at least one such vertex exists from the previous part of the question. But then

$$\deg_G(u) + \deg_G(v) = 3 + \deg_G(v) \geq 3 + 4$$

where the inequality follows because $v \in \mathcal{H}$ and every vertex in $\mathcal{H}$ has degree 4 or more. This inequality implies that the edge (u, v) will be present in $[G]$. Thus the vertices with degree 3 in G will have degree at least 4 in $[G]$. Further, one or more of the vertices in $\mathcal{H}$ will have higher degree in $[G]$: there will either be four vertices of degree 5 (if the two new edges involve G's vertices of degree 4) or at least one vertex of degree 6.

Call the graph produced by adding the two edges discussed above $G'(V, E')$. In light of the discussion above, the vertices in $\mathcal{H}$ could have degrees

$$(4, 5, 5, 6), (5, 5, 5, 5) \text{ or } (4, 4, 6, 6) \tag{8.2}$$

in G'. And if a vertex has degree 6, it must be adjacent to every other vertex in G', as $|V| = 7$.

Now consider the vertex—call it w—that has degree 2 in G. Suppose that, after the addition of the two edges discussed above, either one of the first two possibilities in Eqn. (8.2) holds. Then reasoning similar to that used above establishes that there is a vertex s of degree 5 in G' that is not adjacent to w. As

$$\deg_{G'}(w) + \deg_{G'}(s) = 2 + 5 \geq 7,$$

the closure construction must eventually add the edge (w, v), which implies $\deg_{[G]}(w) \geq 2$ and so we will have proven that every vertex in $\mathcal{L}$ has higher degree in $[G]$ than in G.

All that remains is to deal with the possibility that the vertices in $\mathcal{H}$ could have degrees $(4, 4, 6, 6)$ in G'. If this happens, the degree sequence of G' is

$(2, 4, 4, 4, 4, 6, 6)$. Consider a vertex of degree 4. Two of its four neighbours must be the vertices of degree 6, as each of them is connected to every other vertex in the graph. Thus the vertices of degree 4 have only two other neighbours. Given that there are 4 vertices with degree 4, there must be at least one pair of them that are not adjacent. The closure construction will add an edge between them, producing a new graph, G'' whose degree sequence is $(2, 4, 4, 5, 5, 6, 6)$. Then reasoning similar to that above shows that the closure will include an edge between w, the vertex of degree two, and the vertices of degree 5. This establishes that every vertex in $\mathcal{L}$ has higher degree in $[G]$ than it does in G.

(c) As every vertex in $\mathcal{L}$ has higher degree in $[G]$ than it does in G, we know:

- Every vertex $v \in V$ has $\deg_{[G]}(v) \geq 3$.
- $[G]$ has at most one vertex with degree 3, and every other vertex has degree 4 or more.

Ore's theorem then tells us that $[G]$ is Hamiltonian and then, by Bondy-Chvátal, G must be Hamiltonian too.

(9) (after Jungnickel's Exercise 1.4.5).
I will give two solutions to this problem, though both are summarized by the figure below, which shows that a graph on six vertices must contain at least eight edges if its closure is to be the complete graph K_6.

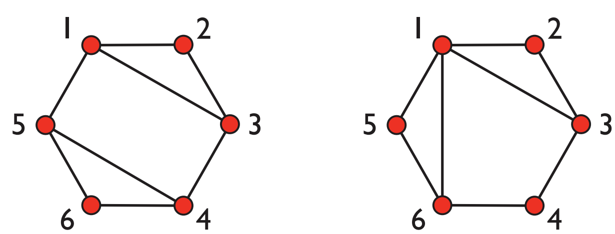


Figure 1: *Two ways to add a pair of edges to a six-vertex cycle so that the resulting graph will have K_6 as its closure.*

Using Hamiltonian cycles
We know that the complete graph K_6 is Hamiltonian, so any graph G that has $[G] = K_6$ must also be Hamiltonian and so must include a six-vertex cycle. Now clearly we need at least one more edge as each vertex in a cycle has degree two, so that any pair of non-adjacent vertices have degrees summing to four, which is insufficient to force the addition of a new edge during the closure construction. Instead, we must have at least two vertices with degree three or more. One can produce a pair of vertices of degree three by adding a single edge to the six-cycle, but the degree-three vertices produced in this way are already adjacent, so a seven-edge graph produced in this way is its own closure. To get a graph whose closure is

different from itself we must then add two edges to the six-cycle. We can do this in such a way as to produce four vertices of degree three (leftmost example in Figure 1) or two vertices of degree three and one of degree four (rightmost example).

Directly, from definitions
It's not hard to prove the following sequence of propositions:

(i) If a vertex v in a graph G has $\deg(v) = 1$, it does not acquire any new edges during the construction of $[G]$.

(ii) If the closure of a graph G on six vertices is $[G] = K_6$, each vertex v of G must have $\deg v \geq 2$, so G must have at least six edges.

(iii) If every vertex in a graph has degree two, the graph consists of a union of cycles.

(iv) If a graph G has more than one connected component then $[G] = G$.

(v) If the closure of a graph G on six vertices is $[G] = K_6$, G must contain a six-vertex cycle.

After this, the argument is the same as the one above.