MATH20902: Discrete Maths, Solutions to Problem Set 6

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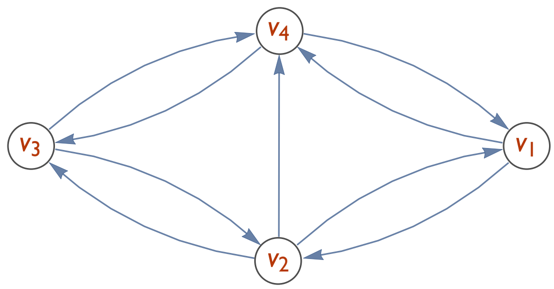
(1). The digraph in question is



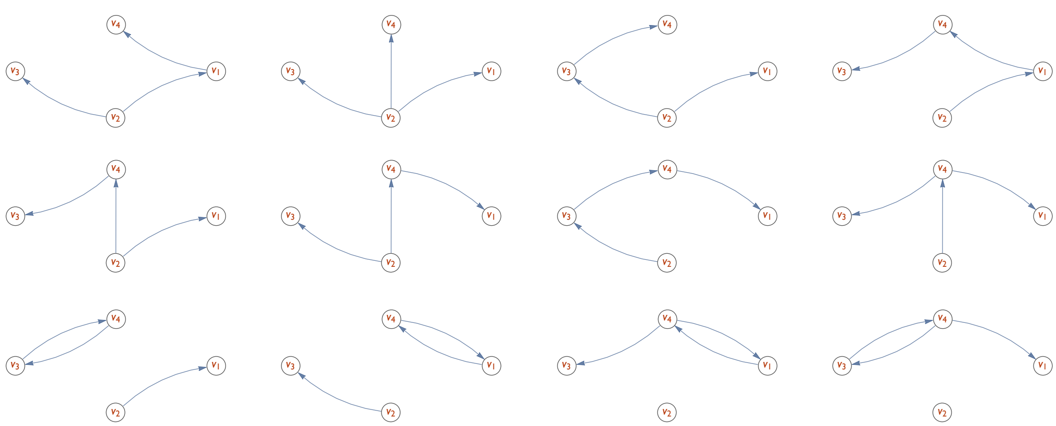
(a) If $G(V, E)$ is a digraph then number of spregs with distinguished vertex v is given by

$$\prod_{u \in V,\, u \neq v} \deg_{in}(u)$$

since this is the number of distinct choices for the predecessors of the vertices $u \neq v$. In the graph above, for spregs with distinguished vertex $v = v_2$, this yields

$$\prod_{u \in V,\, u \neq v_2} \deg_{in}(u) \;=\; \deg_{in}(v_1) \times \deg_{in}(v_3) \times \deg_{in}(v_4) \;=\; 2 \times 2 \times 3 \;=\; 12.$$

(b) The spregs are illustrated below. All except those in the bottom row (which contain cycles) are spanning arborescences.

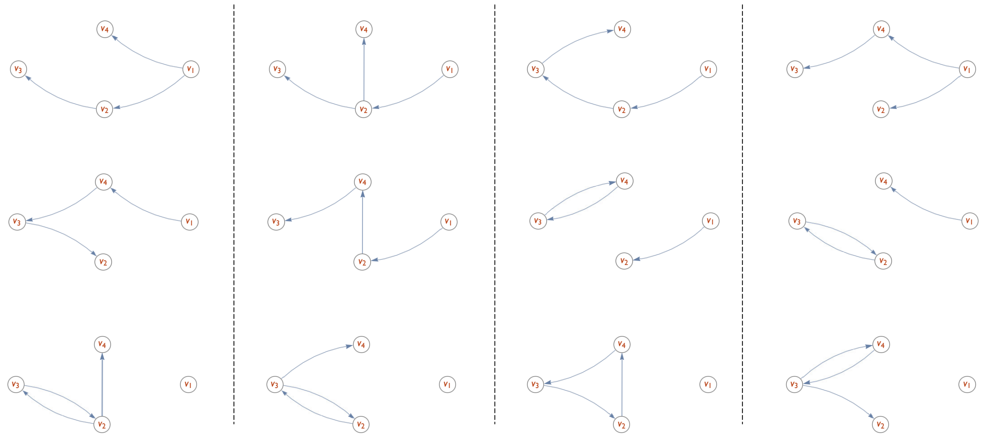


(c) This part of the problem touches on the details of the proof of the Tutte's Matrix Tree Theorem.

- Reasoning similar to that used in part 1 says that there are

$$\prod_{u \in V,\, u \neq v_1} \deg_{in}(u) \;=\; \deg_{in}(v_2) \times \deg_{in}(v_3) \times \deg_{in}(v_4) \;=\; 2 \times 2 \times 3 \;=\; 12$$

  spregs with distinguished vertex v_1: they are illustrated below.



- By inspection of the graphs above, there is just a single spreg containing the cycle (v_2, v_4, v_3, v_2): it's in the bottom row, second-from-right.
- Vertices v_2–v_4 correspond to rows 1–3, respectively, of the matrix $\hat{L}_1$ and so the element of S_3 that corresponds to the directed cycle (v_2, v_4, v_3, v_2) is $\sigma = (1, 3, 2)$.

(2). It's helpful to recall the definition of a spanning arborescence rooted at v, which is equivalent to the following:

Definition. *A digraph $T(V, E)$ is a* ***spanning arborescence rooted at*** *v if:*

- *Every vertex $u \in V$ is reachable from v.*
- *The undirected graph one obtains by ignoring the directedness of the edges is a tree.*

First note that the two possibilities in the lemma are mutually exclusive: a graph that contains a directed cycle cannot be a spanning arborescence as, when we ignore the directedness of its edges, we get an undirected graph containing a cycle and such a graph is, by definition, not a tree (as a tree is a connected, acyclic graph).

Now to prove the lemma, consider the following process. Choose an arbitrary vertex $u \neq v$ in the spreg. It has a unique predecessor (this is the definition of a spreg) which we can call u'. Similarly, u' has a unique predecessor, which we could call u''. We can continue in this way, following edges backward through the spreg until one of two things happen:

(a) we reach a vertex that we've already visited;

(b) we reach the distinguished vertex v, which has no predecessor.

And as the graph has only finitely many vertices, one of these two possibilities *must* occur.

If outcome 1 occurs for some starting vertex u, then we have found a directed cycle and so the undirected version of the spreg contains a cycle, meaning that it is not a tree, so the spreg is not a spanning arborescence. Thus to finish the proof we need to show that if outcome 2 occurs for every starting vertex $u \neq v$, then the spreg is a spanning arborescence rooted at v.

And if outcome 2 *does* occur for every vertex $u \neq v$, then clearly every vertex is reachable from v, for the process described above traces in reverse over all the required paths. This observation also establishes that if we ignore the directedness of the edges, the resulting graph is connected, since each vertex $u \neq v$ is clearly connected to v and, through v, to every other vertex in the spreg. Finally, the undirected version of the spreg must also be acyclic, for if it contained a cycle there would need to be either a directed cycle in the spreg (in which case outcome 1 would have occurred for those vertices in the cycle) or at least one vertex with in-degree two or more, a possibility ruled-out by the definition of spregs. Thus if outcome 2 occurs for every vertex $u \neq v$ the spreg is a spanning arborescence and we are finished.