MATH20902: Discrete Maths, Solutions to Problem Set 8

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(1) (After Jungnickel's exercise 1.5.10).
The first of the two parts is easier, but the second is more interesting.

(a) We know that in a planar graph with n vertices, m edges and girth g, the following inequality must hold:

$$m \leq \frac{g(n-2)}{g-2}$$

Now, the Petersen graph has $n = 10$ and its girth (the length of the shortest cycle) is $g = 5$, so the inequality above tells us that if it is planar, it can have, at most,

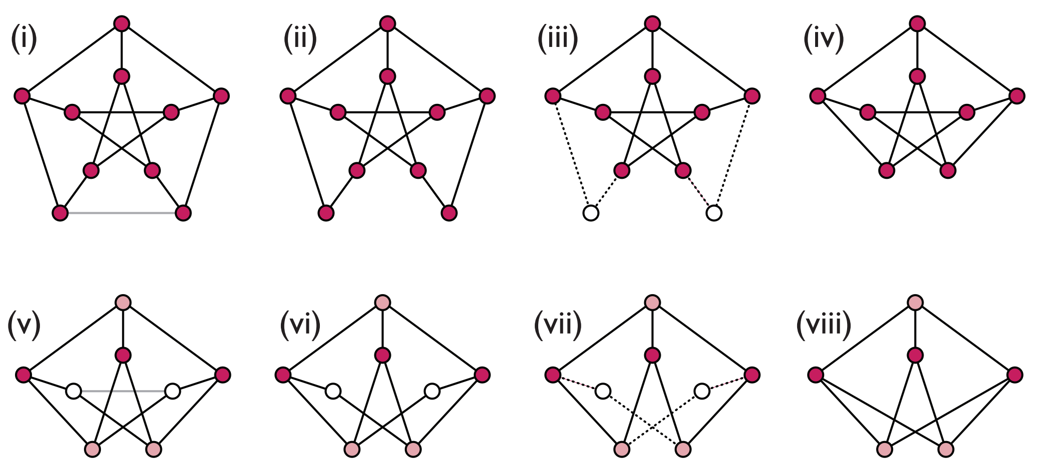
$$\frac{g(n-2)}{g-2} = \frac{5 \times (10-2)}{5-2} = \frac{5 \times 8}{3} = \frac{40}{3}$$

edges. But $(40/3) < (42/3) = 14$ while the actual graph has 15 edges: that's too many and thus the Petersen graph cannot be planar.

(b) To use Kuratowski's Theorem to prove that the Petersen graph is non-planar, we must find a subgraph that is homeomorphic to either $K_{3,3}$ or K_5. Let's call the Petersen graph itself G. Looking at the diagram of G, it's clear that the graph cannot be produced directly by subdivision of a graph with fewer vertices: the basic step of subdivision inserts vertices of degree 2 and all the vertices in G have degree 3.

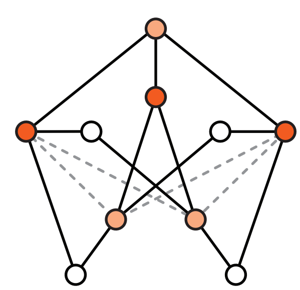
On the other hand, G has 10 vertices, while $K_{3,3}$ has 6 and K_5 has only 5: thus we need to get rid of some vertices, so we should try looking at subgraphs of G (produced by deleting edges) and hope that some of them are more obviously homeomorphic to either $K_{3,3}$ or K_5. Further thought shows that K_5 is not an option: the process of subdivision never increases the degree of any vertex, and all the vertices in the Petersen graph have degree 3, while all those in K_5 have degree 4. We should be looking for a subgraph homeomorphic to $K_{3,3}$.

This requires a bit of trial and error, but not a great deal: the idea is to work backward through the process of subdivision. The figure below shows the sequence of steps that I thought through while preparing this solution.

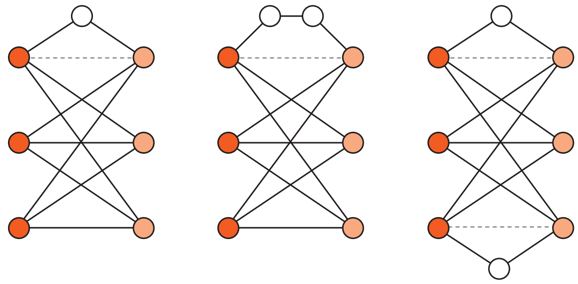


- As a first step, look what happens when we consider a subgraph of G with just one less edge: this is illustrated in parts (i) and (ii) of the sequence above.
- The new graph has two vertices with degree 2, so we may imagine these to have been inserted during the process of subdivision: steps (iii) and (iv) show the results of undoing these insertions. We are left with a graph with two fewer vertices.
- The result of the previous step is, again, a graph in which every vertex had degree three. But then, as we saw above, we should be able to eliminate two vertices, by first removing a suitably-chosen edge. Using this approach, we can get down to 6 vertices, the same number as $K_{3,3}$ has. Panel (v) shows one possible solution: if we had deleted the edge shown in grey from G, then the two vertices at its endpoints would be left with degree 2 and we could again regard them as insertions made during subdivision, and so remove them. This is illustrated in the sequence (vi)–(vii).
- All the figures in the bottom row, (v)–(vii), are shaded in such a way as to highlight the bipartite division of the final graph, (viii). It has 6 vertices, all of degree three, that fall into two disjoint groups. Each vertex in one group is connected to each vertex in the other and there are no intra-group edges. In short, (viii) is a representation of $K_{3,3}$.

Putting all this reasoning together, one can draw the figure below, which is a subgraph of the Petersen graph that is homeomorphic to $K_{3,3}$. Here the plotting conventions are as in the answer to Problem (2): the vertices of $K_{3,3}$ are shown in two shades of orange while those added during the four rounds of subdivision are shown in white. Edges removed during subdivision are shown with pale grey dashes.



(2). There are (infinitely!) many correct answers to this question, but the figure below shows one particularly simple group. The three diagrams show subdivisions of $K_{3,3}$ in which deleted edges appear as pale, dashed grey line segments and the vertices added during subdivision are shown in white, while those inherited from $K_{3,3}$ are shown in shades of orange.



It's clear that the leftmost graph is not isomorphic to the other two, as it has fewer vertices. The middle and rightmost graphs are not isomorphic either, for while both contain a pair of (new) vertices of degree two, these two are adjacent in the middle graph, but not in the rightmost one.

The table below summarizes the sizes of the edge and vertex sets

Graph	$\|V\|$	$\|E\|$	$\|E\| - \|V\|$
Left	7	10	3
Middle	8	11	3
Right	8	11	3

(3) (After Jungnickel's exercise 1.5.7)**.**
Consider a graph G with vertex set V and edge set E. The basic step in making a subdivision of G is to replace an edge, say, $e = (a, b)$ with a path

$$(a, x_1, \dots, x_k, b)$$

where the $x_1, \dots, x_k$ are an arbitrary number k of new vertices. This process produces a new graph—call it G' with vertex set V' and edge set E'—and it is clear from the construction that

$$|V'| = |V| + k \qquad \text{and} \qquad |E'| = |E| - 1 + (k+1).$$

But this implies

$$\begin{aligned} |E'| - |V'| &= (|E| - 1 + (k+1)) - (|V| + k) \\ &= (|E| + k) - |V| - k \\ &= |E| - |V|, \end{aligned}$$

which is just the thing we were asked to show. A very scrupulous reader could use this observation as the basis of an inductive proof where the induction is on the number of edges replaced during the construction of the subdivision.

(4) (After Jungnickel's exercise 1.5.14)**.**
When approaching a problem like this, a good way to start is to write down everything you know. In this case, as G is a planar graph with $n \geq 3$ vertices, we have the following bound on the number of edges:

$$m \leq 3n - 6. \tag{4.1}$$

Of course, we also have a potentially sharper bound that involves the girth of G, but this problem says nothing about girth. In addition to the inequality above, we have one other fact, which is true of all graphs. If G has vertex set V, we can use the Handshaking Lemma to write

$$\sum_{v \in V} \deg(v) = 2m \qquad \text{or} \qquad \frac{1}{2} \sum_{v \in V} \deg(v) = m. \tag{4.2}$$

As problem asks about n_d, the number of vertices whose degree is less than or equal to d, it will prove convenient to have a notation for the number of vertices whose degree is exactly j. Let's define l_j to be

$$l_j = |\{v \in V \,|\, \deg(v) = j\}|\,.$$

It's now easy to write down formulae for n_d and for the total number of vertices, n:

$$n_d = \sum_{j=0}^{d} l_j \qquad \text{and} \qquad n = \sum_{j=0}^{n-1} l_j.$$

We can also use the l_j to rewrite the relation in (4.2), which becomes

$$m = \frac{1}{2} \sum_{j=0}^{n-1} j\, l_j.$$

Putting this result together with (4.1) yields

$$3n - 6 \;\geq\; \frac{1}{2} \sum_{j=0}^{n-1} j\, l_j. \qquad \text{or} \qquad \sum_{j=0}^{n-1} j\, l_j \;\leq\; 6n - 12, \tag{4.3}$$

which is beginning to look a bit like the thing we are trying to prove. To complete the argument we need to look more closely at the sum in the expressions above. The first step is to break the sum into two pieces: one involving the vertices of degree no more than d and the other for those of higher degree:

$$\sum_{j=0}^{n-1} j\, l_j = \left(\sum_{j=0}^{d} j\, l_j \right) + \left(\sum_{j=(d+1)}^{n-1} j\, l_j \right). \tag{4.4}$$

Now we can obtain a lower bound on these sums by replacing the factor of j in each with its smallest value:

$$\begin{aligned}
\left(\sum_{j=0}^{d} j\, l_j \right) + \left(\sum_{j=(d+1)}^{n-1} j\, l_j \right) \;&\geq\; \left(\sum_{j=0}^{d} 0 \times l_j \right) + \left(\sum_{j=(d+1)}^{n-1} (d+1) \times l_j \right) \\
&\geq\; (d+1) \sum_{j=(d+1)}^{n-1} l_j \\
&\geq\; (d+1)(n - n_d), \qquad (4.5)
\end{aligned}$$

where the last line follows because the sum $\sum_{j=(d+1)}^{n-1} l_j$ counts those vertices whose degree exceeds d and, as there are n vertices in total, there are exactly $(n - n_d)$ such high-degree vertices.

Finally, putting Eqns (4.3)–(4.5) together, we have

$$(d+1)(n - n_d) \;\leq\; \left(\sum_{j=0}^{d} j\, l_j \right) + \left(\sum_{j=(d+1)}^{n-1} j\, l_j \right) \;\leq\; 6n - 12$$

or, tidying things up,

$$\begin{aligned}
(d+1)(n - n_d) &\leq 6n - 12 \\
nd + n - (d+1)n_d &\leq 6n - 12 \\
-(d+1)n_d &\leq 5n - dn - 12 \\
-(d+1)n_d &\leq (5-d)n - 12 \\
(d+1)n_d &\geq (d-5)n + 12 \\
n_d &\geq \frac{(d-5)n + 12}{d+1}
\end{aligned}$$

just as advertised.

(5) (Direct proofs that K_5 and $K_{3,3}$ aren't planar)**.** The main idea needed for this problem is that if a vertex v lies in the interior of some Jordan curve C, while a second vertex u lies in the exterior of C, then any curve that represents an edge connecting u to v must intersect C.

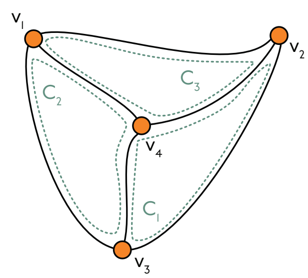


Figure 1: *The three cycles C_1, C_2 and C_3 used in the proof that K_5 can't have a planar diagram.*

(a) The various cycles mentioned in the problem are illustrated in Figure 1: they're defined so that the vertex v_j lies in the exterior of C_j. Thus, for example, $v_1 \in \text{ext}(C_1)$ and $v_3 \in \text{ext}(C_3)$.

Consider the problem of adding the vertex v_5 to the diagram in Figure 1 in such a way as to get a planar diagram for K_5. The point representing v_5 must lie in the exterior of C_1, as otherwise the curve representing the egde (v_1, v_5) would have to cross C_1. Similar arguments show $v_5 \in \text{ext}(C_2)$ and $v_5 \in \text{ext}(C_3)$, so

$$v_5 \in \bigcap_{j=1}^{3} \text{ext}(C_j), \tag{5.1}$$

which is the result we sought.

(b) Now consider the Jordan curve that represents the edges $\{(v_1, v_2),\ (v_2, v_3),\ (v_3, v_1)\}$. On the one hand, $v_4 \in \text{int}(C)$ and, on the other hand, Eqn. (5.1) implies that $v_5 \in \text{ext}(C)$. But then any curve that represents the edge (v_4, v_5) must cross C, which means that it's impossible to find a planar diagram for K_5.

(c) One can prove that $K_{3,3}$ is nonplanar in a similar way. To be concrete, say that the vertex set of $K_{3,3}$ is

$$V = \{u_1, u_2, u_3, v_1, v_2, v_3\}$$

and that the edge set includes all possible edges of the form (u_j, v_k). The left panel of Figure 2 then shows a planar diagram for the cycle that includes all such edges running between vertices u_1, v_1, u_2 and v_2: one can also think of it as a planar diagram for $K_{2,2}$. The right panel of Figure 2 shows the result of adding one more vertex, u_3, and it's clear that, up to renumbering of the

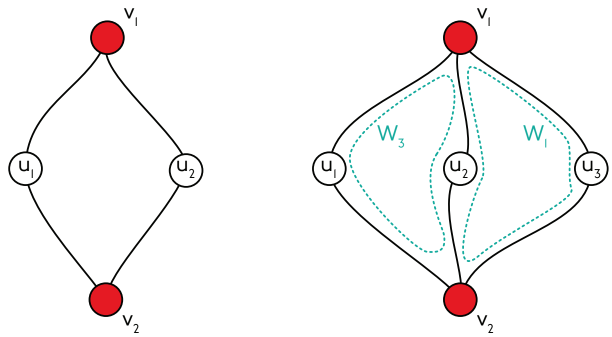


Figure 2: *At left, the cycle* $(u_1, v_1, u_2, v_2, u_1)$ *and at right, a planar diagram for* $K_{3,3}\backslash\{v_3\}$.

vertices, any planar diagram for $K_{3,3}\backslash\{v_3\}$—which is isomorphic to $K_{3,2}$—must look similar.

The right panel of Figure 2 also illustrates two cycles, W_1 and W_3, with the properties that $u_j \in \text{ext}(W_j)$. Now think about adding v_3 to the diagram: the point representing v_3 must be in the exterior of W_1 (so that we can draw the edge (u_1, v_3) without crossing any other edges) and, for similar reasons involving u_3 and the edge (u_3, v_3), the point representing v_3 must also lie in the exterior of W_3. Thus the analogue of Eqn. (5.1) is

$$v_3 \in \text{ext}(W_1) \bigcap \text{ext}(W_3).$$

Finally, consider the Jordan curve W that represents the cycle $(u_1, v_1, u_3, v_2, u_1)$. The reasoning in the preceeding paragraph says $v_3 \in \text{ext}(W)$, but $u_2 \in \text{int}(W)$, so any curve representing the edge (u_2, v_3) must cross W, which implies that $K_{3,3}$ can't have a planar diagram.

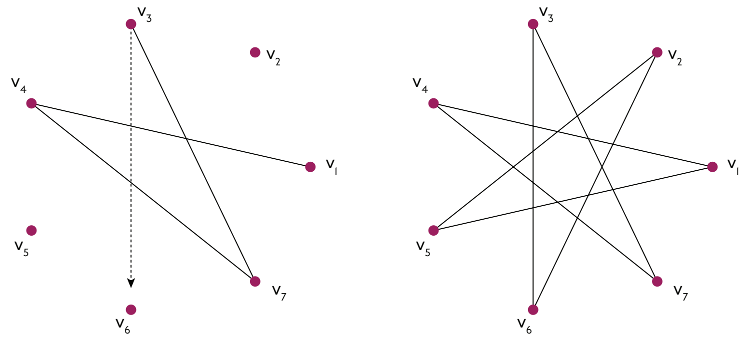


Figure 1: *Two stages in the construction of a thrackle embedding for C_7: all edges are of the form (v_j, v_{j+3}).*

(6) (Conway's thrackles)**.** I learned about thrackles from Bondy and Murty's book, which includes more exercises about them.

(a) The standard drawing for K_3—as a triangle with the vertices on the corners—is a thrackle embedding: each edge shares exactly one point, an endpoint, with each of the other two edges.

(b) The thrackles for odd-length cycles are a generalization of the five-pointed star pictured in the problem set. For a general k-cycle with k an odd number you can prepare a star-shaped thrackle embedding as follows:

- Define a set of angles $\theta_j = 2\pi(j-1)/k$ (with $1 \le j \le k$) and draw points v_j on the unit circle at positions

$$v_j = (x_j,\, y_j) = (\cos(\theta_j),\, \sin(\theta_j)).$$

- As k is an odd number, we can write it as $k = 2m+1$ with $m \in \mathbb{N}$. Draw a line segment from v_1 to v_{1+m}, then another from v_{1+m} to v_{1+2m}, and so on, adding all possible line segments connecting points v_{1+jm} to $v_{1+(j+1)m}$. The subscripts here should be interpreted periodically so, for example, $v_0 = v_k$ and $v_{k+1} = v_1$. Figure 1 illustrates this construction for the case $k = 7$.

To prove, conclusively, that this construction leads to a thrackle embedding for C_{2m+1} we have to do two things:

i) establish that the edges described above really do link up to form a cycle of exactly k vertices;

ii) establish that all pairs of edges intersect each other exactly once, either at their endpoints of somewhere in their interiors.

For the first of these, consider the sequence of vertices

$$(v_1, v_{1+m}, v_{1+2m}, \ldots, v_{1+km}). \tag{6.1}$$

This is the order in which our construction visits the v_j and it's easy to see that when the vertex numbers are considered periodically, $v_{1+km} = v_1$, so the sequence starts and ends with the same vertex. If, additionally, we can show that all the vertices in between are distinct, then we've proven that (6.1) describes a cycle isomorphic to C_k. So then, suppose that there are two vertices $v_{1+j_1 m}$ and $v_{1+j_2 m}$, with $1 < j_1, j_2 < k$ that actually correspond to the same point in the diagram. Then the subscripts $(1 + j_1 m)$ must $(1 + j_2 m)$ differ by a multiple of $k = 2m + 1$, so

$$(1 + j_1 m) - (1 + j_2 m) \;=\; (j_1 - j_2)m \;=\; 0 \bmod k.$$

But this means $\exists\, b \in \mathbb{Z}$ such that $(j_1 - j_2)m = bk$, which means

$$(j_1 - j_2)m \;=\; b(2m+1) \qquad \text{and} \qquad b = (j_1 - j_2)\left(\frac{m}{2m+1}\right). \tag{6.2}$$

It's clear that the second factor, $m/(2m + 1)$, is a rational number in lowest terms (that is, its numerator and denominator have no common factors), so the only ways that (6.2) can be true are if $b = 0$, so $j_1 = j_2$, contradiciting our assumption that the vertices were distinct, or if b is a nonzero multiple of $k = (2m + 1)$, contradicting the fact that $1 < j_1, j_2 < k$. This means that the only repeated vertex in the sequence (6.1) is $v_1 = v_{1+km}$ and so the sequence does, indeed, specify a cycle isomorphic to C_k.

To check the intersection properties of the edges, consider the edge (v_1, v_{1+m}): the symmetry of the diagram means that if this edge behaves correctly, then so do all the others. This edge meets two others, (v_{1+m}, v_{1+2m}) and $(v_{1+(k-1)m}, v_1)$, at their shared end points and so should interesect the remaining $k - 3$ edges in their interiors. Consider the vertices

$$\{v_2, v_3, \ldots, v_m\}.$$

There are $(m - 1) = (k - 3)/2$ of them and they all lie on the arc of the unit circle that runs counter-clockwise from v_1 to v_m. Each of these vertices has degree two: each vertex v_j with $2 \le j \le m$ is connected to both v_{j+m} and v_{j-m}. And both v_{j+m} and $v_{j-m} = v_{j+m+1}$ lie in that arc of the unit circle which runs counter-clockwise from v_{m+1} back to v_1. Thus all $2 \times (m-1) = 2m-2 = k-3$ of these edges intersect the edge (v_1, v_{1+m}) exactly once, which is what we sought to prove.

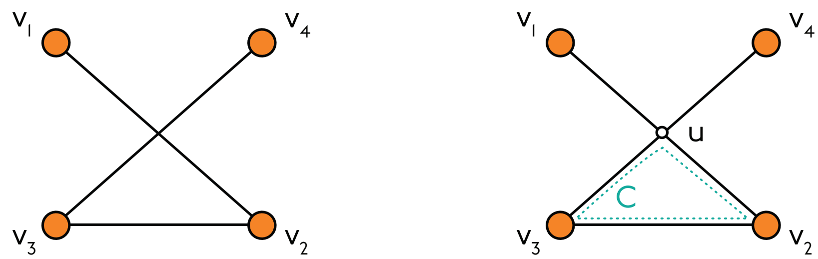


Figure 2: *Any thrackle embedding for* $C_4 \backslash (v_1, v_4)$ *must look similar to the left panel. In the right panle, the point* u *is the intersection of the arcs representing the edges* (v_1, v_2) *and* (v_3, v_4)*, while* C *is the Jordan curve formed by the union of the arcs* (u, v_2), (v_2, v_3) *and* (v_3, u)*.*

(c) Consider Figure 2: it illustrates the main point of a proof by contradiction that C_4 cannot have a thrackle embedding. If we had such an embedding, then deleting the curve representing the edge (v_1, v_4) must yield a result similar the figure's left panel: the curves representing (v_1, v_2) and (v_3, v_4) must intersect exactly once. Define u to be this point of intersection, then consdier the Jordan curve $C = (u, v_2, v_3, u)$ illustrated in the left panel of Figure 2. Both v_1 and v_4 lie in the exterior of C and this means that any curve that connects them must cross C an even number of times (which includes the possibility of zero crossings). This precludes a thrackle embedding of C_4, for in any such embedding the curve representing (v_1, v_4) must cross the one representing (v_2, v_3) exactly once.