MATH20902: Discrete Maths, Solutions to Problem Set 5

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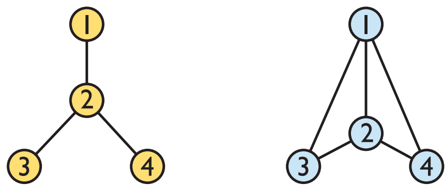
(1) If one numbers the vertices as shown below



the corresponding Laplacians are given by $L = D - A$ where A is the adjacency matrix and D is a diagonal matrix with $D_{jj} = \deg(v_j)$. For the graphs above this leads to

$$L_{left} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad L_{right} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}.$$

The graph on the left *is* a tree, so we expect it to contain only a single spanning tree consisting of the graph itself. And, reassuringly, if we apply Kirchoff's Matrix-Tree Theorem the matrix $\hat{L}_2$ formed by deleting the second row and column of L_{left} we find

$$\det(\hat{L}_2) \;=\; \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \;=\; 1.$$

By way of contrast, deleting the first row and column of L_{right} leads to

$$\begin{aligned} \det(\hat{L}_1) &= \det \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} \\ &= 3 \times \det \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - (-1) \times \det \begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix} + (-1) \times \det \begin{bmatrix} -1 & 2 \\ -1 & 0 \end{bmatrix} \\ &= 12 - 2 - 2 \\ &= 8. \end{aligned}$$

The eight spanning trees are illustrated in Figure 1.

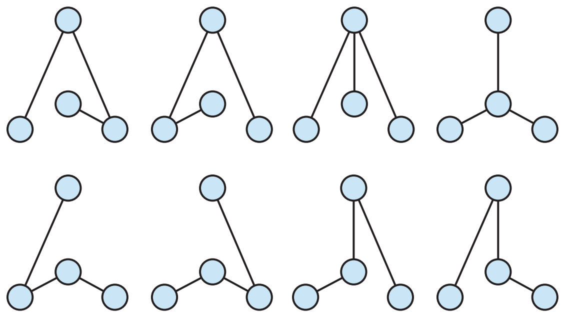
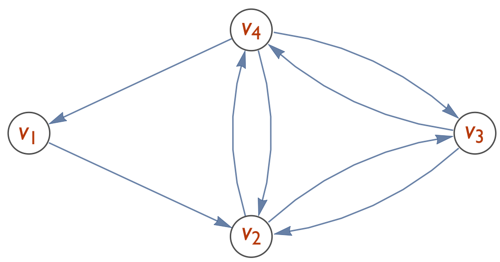


Figure 1: *The eight spanning trees for the graph at right in Problem (1).*

(2) If one numbers the vertices as shown in the problem



the corresponding Laplacian is $L = D - A$ where A is the adjacency matrix and D is a diagonal matrix with $D_{jj} = \deg_{in}(v_j)$. For the graph above this is

$$L = D - A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 2 \end{bmatrix}.$$

Tutte's Matrix-Tree Theorem then says that numbers of spanning arborescences rooted at v_j is given by $\det(\hat{L}_j)$, where $\hat{L}_j$ is the matrix formed by deleting the j-th row and column of L. The results for the Laplacian above are listed in Table 1.

(3) The permutation group S_6 has $6! = 720$ elements and so it is unlikely that your choice of four random elements includes any of those listed in Table 2, but the basic principles are the same: Figure 2 shows the digraphs I used to compute the cycle decompositions and sets $\text{fix}(\sigma)$, while the values of $\text{sgn}(\sigma)$ come from the rules

- If σ is the identity permutation, then $\text{sgn}(\sigma) = 1$.
- If σ is a cycle of length l then $\text{sgn}(\sigma) = (-1)^{l+1}$.
- If σ has a decomposition into $k \geq 2$ disjoint cycles whose lengths are $l_1, \ldots, l_k$ then

$$\text{sgn}(\sigma) = (-1)^{L+k} \qquad \text{where} \qquad L = \sum_{j=1}^{k} l_j.$$

v_j	$\hat{L}_j$	$\det(\hat{L}_j)$	v_j	$\hat{L}_j$	$\det(\hat{L}_j)$
v_1	$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$	3	v_3	$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix}$	4
v_2	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$	3	v_4	$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$	5

Table 1: *The number of spanning arborescences rooted at each of the vertices in the digraph from Problem (2).*

σ	$\text{fix}(\sigma)$	Cycle decomposition	$\text{sgn}(\sigma)$
$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 2 & 6 & 1 \end{pmatrix}$	$\emptyset$	$(1,3,5,6)(2,4)$	1
$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 4 & 6 & 5 & 1 \end{pmatrix}$	$\{2,5\}$	$(1,3,4,6)$	-1
$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 3 & 4 & 6 & 2 \end{pmatrix}$	$\{3,4\}$	$(1,5,6,2)$	-1
$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 3 & 4 & 1 & 2 \end{pmatrix}$	$\{3,4\}$	$(1,5)(2,6)$	1

Table 2: *The sets* $\text{fix}(\sigma)$*, cycle decompositions and signs* $\text{sgn}(\sigma)$ *for four randomly-selected members of* S_6*.*

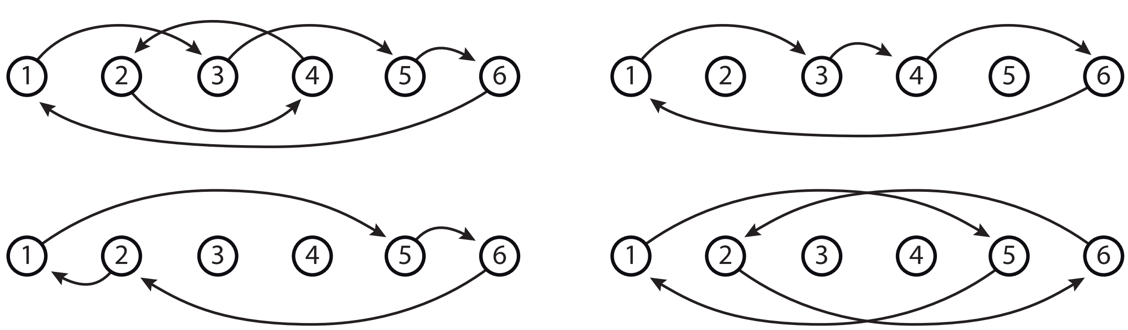


Figure 2: *The digraphs used to compute the cycle decompositions and sets* $\text{fix}(\sigma)$ *for the permutations listed in Table 2.*

(4) Applying the Principle of Inclusion/Exclusion to a union of k sets requires the consideration of 2^k intersections, so this problem, with $k = 4$, is near the limit of what's convenient to do by hand.

(a) It's easiest to prove this by contradiction. Suppose n is a composite number and that all its prime factors exceed $\sqrt{n}$. As n is composite, it must have two or more prime factors. We'll arrange them in ascending order:

$$n = p_1 \times p_2 \times \dots \qquad \text{with, by assumption,} \qquad \sqrt{n} < p_1 \leq p_2 \leq \dots$$

Note that the expression above includes the possibility that $p_1 = p_2$.

Then, on the one hand, we know that

$$n \geq p_1 \times p_2$$

because n may have other prime factors in addition to p_1 and p_2. But on the other hand, as we have assumed that $p_1 > \sqrt{n}$ and $p_2 > \sqrt{n}$ we also have

$$p_1 \times p_2 > (\sqrt{n})(\sqrt{n}) \qquad \text{or} \qquad p_1 \times p_2 > n.$$

This is a contradiction and the only possible escape is to conclude that n's smallest prime factor must satisfy $p_1 \leq \sqrt{n}$.

(b) If we take the universal set to be

$$U = \{2, 3, \dots, 120\}$$

and the subsets X_1, X_2, X_3 and X_4 to be, respectively, the multiples of 2, 3, 5 and 7 then the cardinalities of the various intersections we need are:

Set	Description	Cardinality
X_1	multiples of 2	60
X_2	multiples of 3	40
X_3	multiples of 5	24
X_4	multiples of 7	17
$X_1 \cap X_2$	multiples of 6	20
$X_1 \cap X_3$	multiples of 10	12
$X_1 \cap X_4$	multiples of 14	8
$X_2 \cap X_3$	multiples of 15	8
$X_2 \cap X_4$	multiples of 21	5
$X_3 \cap X_4$	multiples of 35	3
$X_1 \cap X_2 \cap X_3$	multiples of 30	4
$X_1 \cap X_2 \cap X_4$	multiples of 42	2
$X_1 \cap X_3 \cap X_4$	multiples of 70	1
$X_2 \cap X_3 \cap X_4$	multiples of 105	1
$X_1 \cap X_2 \cap X_3 \cap X_4$	multiples of 210	0

This leads to the conclusion that the number of integers in the specified range that are multiples of 2, 3, 5 or 7 is

$$
\begin{aligned}
|X_1 \cup X_2 \cup X_3 \cup X_4| &= \sum_{I \subseteq \{1,\ldots,4\},\, I \neq \emptyset} (-1)^{|I|+1} \left| \bigcap_{j \in I} X_j \right| \\
&= (60 + 40 + 24 + 17) - (10 + 12 + 8 + 8 + 5 + 3) \\
&\quad + (4 + 2 + 1 + 1) - 0 \\
&= 141 - 56 + 8 \\
&= 93
\end{aligned}
$$

(c) Given that $|U| = 119$, the result from part (b) means that there are $119 - 93 = 26$ members of U that are coprime to 2, 3, 5 and 7. And in light of our result from part (a), any composite number in U has a prime factor less than $\sqrt{120} < \sqrt{121} = 11$, so must be divisible by at least one of 2, 3, 5 and 7 and thus must belong to one of the X_j. Thus all 26 members of $U \backslash X_1 \cup X_2 \cup X_3 \cup X_4$ are prime, as are 2, 3, 5 and 7, for a grand total of $26 + 4 = 30$ primes in the specified range.