MATH20902: Discrete Maths, Solutions to Problem Set 3

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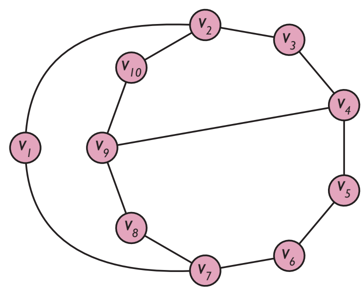
(1). The desired graph contains a cycle of length nine, so one might as well start the construction by drawing that and then add extra edges and/or vertices to create the shorter cycles. A bit of trial-and-error produced the graph illustrated below, which has ten vertices and at least one cycle with each of the lengths five through nine.



The table below gives one example for each of the desired lengths, but is not exhaustive: several examples exist for some of the lengths.

Length	Vertex list for cycle
5	$\{v_2, v_3, v_4, v_9, v_{10}, v_2\}$
6	$\{v_4, v_5, v_6, v_7, v_8, v_9, v_4\}$
7	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_1\}$
8	$\{v_2, v_{10}, v_9, v_4, v_5, v_6, v_7, v_1, v_2\}$
9	$\{v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_2\}$

(2). The first part of the question requires us to check that "is-strongly-connected-to" has the three properties of an equivalence relation.

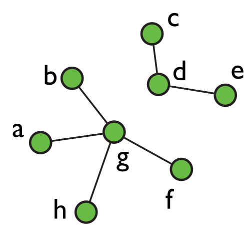
reflexivity True by definition: we say a vertex in a digraph is strongly connected to itself.

symmetry The defintion of strong connectedness requires walks in both directions— a to b and b to a—and so the relation clearly symmetric.

transitivity One can establish this by concatenating walks, as is done for undirected graphs in the lecture notes. If this were an exam question, you would have to write that argument in full.

The strongly connected components of the graph illustrated in the problem are induced by the sets: $\{v_1, v_2, v_3\}$, whose elements all lie on a cycle in the order specified; $\{v_4\}$, which is a strongly connected component on its own and $\{v_5, v_6, v_7\}$, whose elements also lie on a cycle.

(3). The graph below, which has two connected components, provides all the examples we'll need.



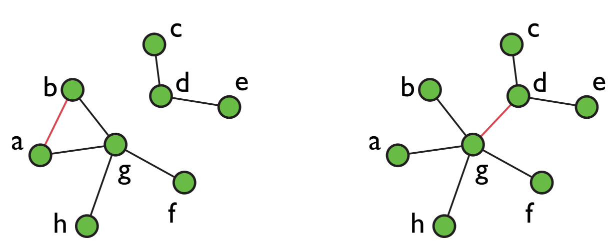
(a) When one adds an edge to a graph there are two possibilities:

- the number of connected components remains unchanged or
- decreases by exactly one.

To see this, first introduce some terms: say that we're adding the edge (u, v) to $G(V, E)$. There are two possibilities:

- the endpoints of the edge u and v lie in the same connected component or
- they lie in the distinct components.

Any change in connectivity must arise from a walk that includes the new edge (all other walks in G are already reflected in its original connected components). And any walk that includes (u, v) must connect a vertex that is connected to u with one connected to v. Say that the connected components containing u and v are, respectively, C_u and C_v. If these two components are the same—if $C_u = C_v$—then the connectivity of the graph remains unchanged, but if u and v lie in distinct components then the addition of the edge (u, v) creates new walks that connect the elements of C_u to those of C_v and so these components are merged in the new graph. Both possibilities are illustrated below, where the new edges are shown in red. At left, the new edge (a, b) joins vertices that were already connected, so the graph still has two connected components, while at right the new edge (g, d) joins vertices in distinct components and so the new graph has only a single component.



(b) Suppose we remove an edge $e = (u, v)$. Before we remove the edge, u and v lie in the same connected component and there are two possibilities:

- every walk that connects u to v includes e or

- there is a walk connecting u to v that doesn't include e. In this case the graph contains a cycle that includes e.

In the first case, removing e breaks the connected component that contained u and$v$ into tow pieces, one containing u and the other containing v. IN the second case, the number of components remains the same. The figure above illustrates both possibilities, where now we imagine removing the red edges.

(4). This problem is about doing arithmetic with asymptotic bounds.

(a) Given that $f_j(n) = O\left(g_j(n)\right)$, we know that there are constants c_1 and c_1' such that, for sufficiently large n,

$$f_1(n) \leq c_1 g_1(n) \qquad \text{and} \qquad f_2(n) \leq c_1' g_2(n).$$

Thus, also for sufficiently large n, we have

$$f_1(n) f_2(n) \leq c_1 g_1(n) f_2(n) \leq c_1 c_1' g_1(n) g_2(n),$$

which is the same thing as saying that $f_1(n)f_2(n) = O\left(g_1(n)g_2(n)\right)$ with constant $c_1'' = c_1 c_1'$.

(b) If, for all sufficiently large n, $g_2(n) > g_1(n)$ then of course

$$\begin{aligned} f_1(n) + f_2(n) &\leq c_1 g_1(n) + c_1' g_2(n) \\ &\leq c_1 g_2(n) + c_1' g_2(n) \\ &\leq (c_1 + c_1') g_2(n) \end{aligned}$$

and $(f_1(n) + f_2(n)) = O(g_2(n))$. Note that one can combine this with the result from part (a) to make an inductive proof that the polynomial in Problem (7) is $O(n^k)$.

(5). Let a and b be any two vertices in the graph. Then the connected component containing a must include a itself, as well as all of a's neighbours, and so must contain at least

$$1 + \deg(a) \;\geq\; 1 + \frac{n-1}{2} \;=\; \frac{n+1}{2}$$

vertices: the same bound holds for the connected component containing b. But this means that the two connected components must each contain slightly more than half the vertices in the graph, and hence must, by the Pigeonhole Principle, overlap (that is, their vertex sets must have a non-empty intersection). And if they overlap, they coincide, as connectedness of vertices is an equivalence relation.

(6). This problem isn't so hard, once you think about it carefully, but it took me a long time the first time I encountered it, perhaps because I had been trained as a physicist, so had different habits of mind. In principle, there is nothing wrong with the argument that a symmetric, transitive relation is reflexive, provided that, for every x, there is some $y \neq x$ such that $x \sim y$. Consider, for example, the following relation on $\mathbb{R}$:

$$x \sim y \qquad \text{if and only if } x \times y > 0.$$

Clearly all positive real numbers are related to each other, and all negative reals are related to each other, but 0 is not related to anything, not even to itself.

(7). Here we prove that the asymptotic growth rate of a polynomial is determined by its highest order term. So then, consider

$$f(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_0. \tag{7.1}$$

with $a_k > 0$. It is possible to show that $f(n) = O(n^k)$ using the two parts of Problem (4), but I'll just demonstrate the result directly. The main idea is that, for large enough n, the highest-order term dominates everything else.

To see this, note that dividing both sides of (7.1) by n^k yields

$$\frac{f(n)}{n^k} = a_k + \left(\frac{a_{k-1}}{n} + \cdots + \frac{a_0}{n^k}\right)$$

As we're interested in large values of n, it's easy to get an upper bound for the term in parentheses as follows:

$$\left(\frac{a_{k-1}}{n} + \cdots + \frac{a_0}{n^k}\right) \;=\; \frac{1}{n}\left(a_{k-1} + \cdots + \frac{a_0}{n^{k-1}}\right) \;\leq\; \frac{1}{n}\sum_{j=0}^{k-1} |a_j|.$$

Now define

$$N_1 = \left\lceil \sum_{j=0}^{k-1} |a_j| \right\rceil. \tag{7.2}$$

For $n \geq N_1$ we have that

$$\frac{f(n)}{n^k} \;\leq\; a_k + \frac{N_1}{n} \;\leq\; (a_k + 1) \qquad \text{or} \qquad f(n) \;\leq\; (a_k+1)n^k.$$

And so, taking $c_1 = (a_k{+}1)$ and N_1 as in Eqn. (7.2), we've proven that $f(n) = O(n^k)$.

It's slightly more fiddly to prove that $f(n) = \Omega(n^k)$, but the main idea is still the same: for sufficiently large n, we can be sure that the ratio of the highest-order term to the lower-order terms is as small as we like. So then, consider the following bound on the lower-order terms

$$\left(\frac{a_{k-1}}{n} + \cdots + \frac{a_0}{n^k}\right) \geq \frac{-1}{n}\sum_{j=0}^{k-1} |a_j|.$$

If we could show that for sufficiently large n, this quantity was definitely no more negative than, say, $a_k/2$, we'd be finished, since then the lower-order terms could never completely cancel off the leading term $a_k n^k$. To see how to prove this, divide through by $a_k/2$:

$$\frac{2}{a_k}\left(\frac{a_{k-1}}{n} + \cdots + \frac{a_0}{n^k}\right) \geq \frac{-2}{a_k n}\sum_{j=0}^{k-1}|a_j|$$

This suggests that if we define N_2 by

$$N_2 = \left\lceil \frac{2}{a_k}\sum_{j=0}^{k-1}|a_j| \right\rceil . \tag{7.3}$$

we can conclude that for $n \geq N_2$,

$$\frac{2}{a_k}\left(\frac{a_{k-1}}{n} + \cdots + \frac{a_0}{n^k}\right) \geq \frac{-N_2}{n} \geq -1.$$

Which is equivalent to

$$\left(\frac{a_{k-1}}{n} + \cdots + \frac{a_0}{n^k}\right) \geq \frac{-a_k}{2}$$

and so,

$$\begin{aligned}\frac{f(n)}{n^k} &= a_k + \left(\frac{a_{k-1}}{n} + \cdots + \frac{a_0}{n^k}\right) \\ &\geq a_k - (a_k/2) \\ &\geq a_k/2.\end{aligned}$$

Multiplying by n^k, then yields

$$\begin{aligned}f(n) &= a_k n^k + \left(a_{k-1}n^{k-1} + \cdots + a_0\right) \\ &\geq (a_k/2)n^k,\end{aligned}$$

so if we take $c_2 = a_k/2$ and N_2 as in Eqn. (7.3), the bound above establishes that $f(n) = \Omega(n^k)$. Finally, if we combine this with our earlier proof that $f(n) = O(n^k)$, we have proven that $f(n) = \Theta(n^k)$.