MATH20902: Discrete Maths, Solutions to Problem Set 2

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(1) (Chromatic numbers for famous graphs)**.** Figure 1 shows examples of optimal colourings (ones that use the minimal number of colours) for each of the families discussed in the question.

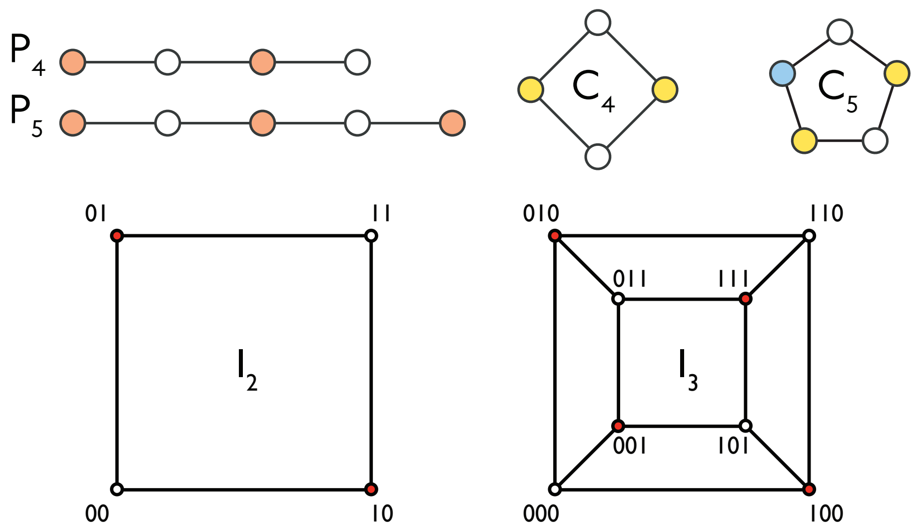


Figure 1: *Examples of optimal colourings for: the path graphs, which have $\chi(P_n) = 2$ for all n; the cycle graphs, which have $\chi(C_n) = 2$ when n is even and $\chi(C_n) = 3$ when n is odd; the cube graphs which have $\chi(I_d) = 2$ for all d.*

(a) For any $n \geq 2$, it's easy to see that $\chi(P_n) = 2$. The definition of P_n tells us that the vertex and edge sets are

$$\begin{aligned} V &= \{v_1, v_2, \ldots, v_n\} \\ E &= \{(v_j, v_{j+1}) \,|\, 1 \leq j < n\}. \end{aligned}$$

Since the edges are all of the form (v_j, v_{j+1}), we know that one of the vertices in the pair will have an even number as its index and the other will have an odd number. This suggests the following colouring

$$\phi_P(v_j) = \begin{cases} 1 & \text{if } j \text{ is even} \\ 2 & \text{if } j \text{ is odd} \end{cases}, \tag{1.1}$$

which establishes that $\chi(P_n) \leq 2$. On the other hand, if $n \geq 2$ then P_n contains at least one edge, hence at least one pair of adjacent vertices and so $\chi(P_n) \geq 2$, which establishes that $\chi(P_n) = 2$.

(b) The cycle graphs C_n have the same vertex sets as the corresponding path graphs P_n. The also have all the same edges, as well as one extra edge, (v_n, v_1). If n is an even number, say, $n = 2m$ with $m \geq 2$, then an argument similar to the one for path graphs above establishes that the chromatic number is $\chi(C_{2m}) = 2$. But when n is odd, say, $n = 2m + 1$ with $m \geq 1$, the edge $(v_n, v_1) = (v_{2m+1}, v_1)$ involves two odd-numbered vertices and thus the map ϕ_P in Eqn. 1.1 fails to be a two-colouring.

One can, however, find a three-colouring,

$$\phi_C(v_j) = \begin{cases} 1 & \text{if } j \text{ is even} \\ 2 & \text{if } j \text{ is odd and } j \neq n \\ 3 & \text{if } j = n. \end{cases} \tag{1.2}$$

which establishes that $\chi(C_{2m+1}) \leq 3$. This inequality, along with the observation that $\chi(C_{2m+1})$ contains at least one edge (and hence a subgraph isomorphic to K_2), means that $\chi(C_{2m+1}) \in \{2, 3\}$ and so if we can prove that $\chi(C_{2m+1}) \neq 2$, we are finished. We can do that as follows: suppose for contradiction that we have a two-colouring $\tilde{\phi}$. Suppose further, without loss of generality, that $\tilde{\phi}(v_1) = 1$. Then, as v_2 is adjacent to v_1, we can conclude that $\tilde{\phi}(v_2) = 2$. Similar reasoning tells us that all odd-numbered vertices v_{2j-1} must have $\tilde{\phi}(v_{2j-1}) = 1$ and all even-numbered vertices v_{2j} must have $\tilde{\phi}(v_{2j}) = 2$. But then, as the number of vertices in the cycle $n = 2m + 1$ is odd, we have

$$\tilde{\phi}(v_1) = \tilde{\phi}(v_n) = 1$$

which, given that v_1 and v_n are adjacent, contradicts our assumption that $\tilde{\phi}$ is a two-colouring. Hence $\chi(C_{2m+1}) > 2$ and so, as we've already proved $\chi(C_{2m+1}) \leq 3$, we've now established that $\chi(C_{2m+1}) = 3$.

(c) The cube graphs I_d all have chromatic number two. To see why, first note that every cube graph contains at least one edge, so $\chi(I_d) \geq 2$. Next, recall that the vertex and edge sets of I_d are

$$\begin{aligned} V &= \{w \mid w \in \{0,1\}^d\} \\ E &= \{(w, w') \mid w \text{ and } w' \text{ differ in a single position}\}. \end{aligned}$$

It's now helpful to introduce the *parity function* $f : \{0,1\}^d \to \{0,1\}$, which is given by the rule

$$f(w) = \begin{cases} 1 & \text{if the string } w \text{ contains an odd number of 1's} \\ 0 & \text{otherwise} \end{cases}$$

Thus, for example,

$$f(000) = 0,\ f(001) = 1,\ f(101) = 0 \text{ and } f(111) = 1.$$

Using this we can specify a two-colouring of the cube graph as follows:

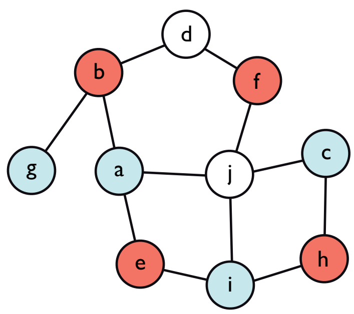
$$\phi_I(w) = 1 + f(w). \tag{1.3}$$

To see that this really does provide a two colouring we need to establish the following proposition:

Proposition. *If the pair (w, w') appears in the edge set of the cube graph then w and w' have opposite parity: $f(w) \neq f(w')$.*

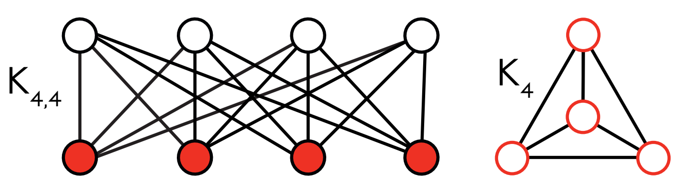
The proof turns on the observation that if the strings w and w' differ in a single position (which is the condition for an edge to connect the corresponding vertices) and if w contains k 1's, then w' must have $k \pm 1$ of them. The point is that the single-position difference between w and w' is either a change $0 \to 1$, which increases the count of 1's by 1, or a change $1 \to 0$, which decreases the count by one. Either way, the parities of w and w' must be different.

(2) (Avoiding clashes)**.** One can solve this problem by constructing a graph whose vertices represent animals and whose edges connect animals that should not be housed together. The chromatic number of this graph is then the minimal number of enclosures and a corresponding colouring tells us how to house the animals: all the animals assigned the same colour can go in the same enclosure. A suitable graph G is



which has been coloured using three colours. On the one hand, this establishes that $\chi(G) \leq 3$, but on the other hand G contains a cycle of odd length—a, b, d, f, j—so we also know $\chi(G) \geq 3$. Thus $\chi(G) = 3$.

(3). The complete graph K_4 can't be a subgraph of the complete bipartite graph $K_{4,4}$. To see why, consider the diagram of $K_{4,4}$ below and note that edges run between all possible pairs of red and white vertices, but that no edges run red-to-red or white-to-white. This observation enables us to prove our result by contradiction. Suppose for contradiction that we can find a subset V' consisting of four vertices from $K_{4,4}$ and that these, along with a suitable subset of the edges, form a subgraph isomorphic to K_4.



As V' contains four vertices and the diagram of $K_{4,4}$ above organises the vertices into only two groups (the ones from the definition of *bipartite*: see the video introducing standard examples of graphs) the Pigeonhole Principle tells us that there must be two vertices in V' that belong to the same group. But then these two cannot be adjacent in any subgraph of $K_{4,4}$, as a bipartite graph has *no* edges running between vertices in the same group. And this contradicts the claim that V' is the vertex set of a subgraph isomorphic to K_4, for in K_4 every vertex is adjacent to every other.

(4). P_4 isn't an induced subgraph of $K_{4,4}$, though C_4 is. Let's say that the vertex set of P_4 is $\{v_1, v_2, v_3, v_4\}$, with edges (v_j, v_{j+1}) for $1 \leq j \leq 3$. Now suppose—aiming at a proof by contradiction—that we have a bijection α mapping these four vertices to a subset of those in $K_{4,4}$. Then $\alpha(v_1)$ and $\alpha(v_3)$ would have to lie in one of the bipartite graph's two subsets (see the previous answer for an illustration of these subsets) while $\alpha(v_2)$ and $\alpha(v_4)$ would lie in the other subset. But then the subgraph of $K_{4,4}$ induced by the vertices $\{\alpha(v_1), \alpha(v_2), \alpha(v_3), \alpha(v_4)\}$ would include the following four edges

$$\{(\alpha(v_1), \alpha(v_2)),\ (\alpha(v_2), \alpha(v_3)),\ (\alpha(v_3), \alpha(v_4)),\ (\alpha(v_4), \alpha(v_1))\}$$

while P_4 has only three edges. Thus the induced subgraph cannot be isomorphic to P_4, which contradicts our initial assumption.

(5). To do these problems, it's enough to exhibit a cycle of the desired length that is contained in the corresponding cube graph.

(a) The vertices of I_2 are labelled by ordered pairs of 1's and 0's and all possible labels occur. Two vertices are adjacent if their labels differ at only a single position. The following cycle is clearly isomorphic to C_4 as, (i) it's a cycle and (ii) it contains 4 distinct vertices.

$$(00,\ 10,\ 11,\ 01,\ 00).$$

In fact, I_2 is isomorphic to C_4.

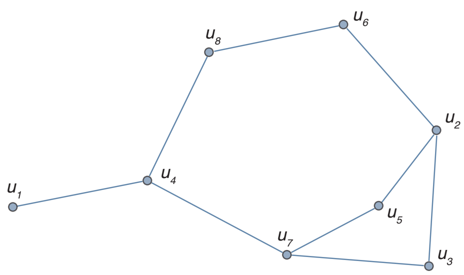
(b) Here a suitable cycle is

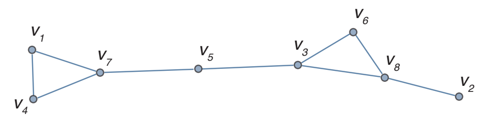
$$(000,\ 100,\ 110,\ 010,\ 011,\ 111,\ 101,\ 001,\ 000).$$

(c) The cycle here is long, but simply related to the one from part (b):

$$(0000,\ 1000,\ 1100,\ 0100,\ 0110,\ 1110,\ 1010,\ 0010,\ 0011,$$
$$1011,\ 1111,\ 0111,\ 0101,\ 1101,\ 1001,\ 0001,\ 0000).$$

(6). If you followed the hint in the question, you should have ended up with diagrams like the two below, which are alternative representations of the graphs in the original illustration.





It is clear that the two graphs are not isomorphic and one way to establish this rigorously is to focus on the vertices with degree three. In the graph at left these are u_2, u_4 and u_7 and two of them, u_4 and u_7 are adjacent. In the graph at right there are also three vertices with degree three—v_3, v_7 and v_8—and two of them, v_3 and v_8, are adjacent. But v_3 and v_8 are also both adjacent to v_6 and there is no corresponding shared neighbour for u_4 and u_7 on the left.

One can use these observations to prove that the graphs aren't isomorphic by contradiction. Suppose there is a suitable bijection α from the vertex set $\{u_1, \ldots, u_8\}$ of the graph on the left to the vertex set $\{v_1, \ldots, v_8\}$ on the right. Then α would have to map the subset $\{u_4, u_7\}$ to the subset $\{v_3, v_8\}$ as these are the adjacent vertices of degree three. But if this bijection really established a graph isomorphism, the existence of the edges (v_3, v_6) and (v_8, v_6) on the right would imply the existence of the edges $(\alpha^{-1}(v_3), \alpha^{-1}(v_6))$ and $(\alpha^{-1}(v_8), \alpha^{-1}(v_6))$ on the left and that would imply the existence of a vertex $\alpha^{-1}(v_6)$ that is adjacent to both u_4 and u_7. No such vertex exists, which contradicts the assumption that α establishes a graph isomorphism and so no such α can exist.

The attentive reader will note that the argument above uses, implicitly, the following proposition, which appeared in lecture:

Proposition. *If $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are isomorphic and $\alpha : V_1 \to V_2$ is the bijection that establishes the isomorphism, then* $\deg(v) = \deg(\alpha(v))$ *for every $v \in V_1$ and* $\deg(u) = \deg(\alpha^{-1}(u))$ *for every $u \in V_2$.*

(7). The basic strategy here is to use the Pigeonhole Principle: we'll show that there are only $(n - 1)$ possible values of $\deg(v)$ and, as there are n vertices, some value must be shared by at least two vertices.

Now, if $G(V, E)$ is a graph with $|V| = n$ vertices then we know that

$$0 \leq \deg(v) \leq (n - 1) \text{ for all } v \in V.$$

That is, a vertex can have a minimum of zero neighbours and a maximum of $(n-1)$. This seems to suggest that there are n possible values of $\deg(v)$, which would spoil our pigeonhole argument. But a moment's thought shows that the maximal and minimal degrees can't occur in the same graph. If some vertex has degree zero, then all the others have degree at most $(n-2)$. Alternatively, if some vertex has degree $(n-1)$ then it must be adjacent to all the others and so they all have degree at least one.

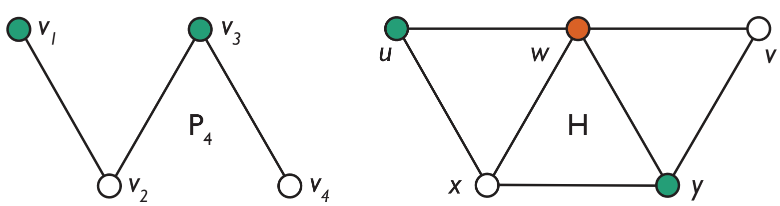
Thus, exactly one of the following three things happens:

- $\deg(v) = 0$ for two or more vertices. In this case the result we seek is clearly true: the entry 0 is repeated in the degree sequence.
- $\deg(v) = 0$ for exactly one vertex and $1 \leq \deg(u) \leq (n-2)$ for all the others. But then there are $(n-1)$ vertices with non-zero degree and only $(n-2)$ possible values for their degrees, so some non-zero value must be repeated in the degree sequence.
- $\deg(v) > 0$ for all $v \in V$. In this case we have $1 \leq \deg(v) \leq (n-1)$ for all n vertices and so, as above, some non-zero value must be repeated in the degree sequence.

(8) (Colouring non-adjacent vertices)**.**
The question asks us to prove the following proposition or find a counterexample.

Proposition. *If $G(V, E)$ is a graph and $k = \chi(G)$ is its chromatic number then, for any two two non-adjacent vertices u, $v \in V$, there exists some k-colouring ϕ such that $\phi(u) = \phi(v)$.*

The proposition is false, as one can show by considering either of the two graphs below. The clearest of the two, P_4 at left, was suggested by Yu Tian, a student in 2017's version of the course, while the graph H at right is an example I made up.



We saw in Problem (1) that P_4 has $\chi(P_4) = 2$ and that any optimal colouring ϕ must assign one colour to vertices v_j with j odd and the other colour to vertices with j even. Thus although v_1 and v_4 are not adjacent, $\phi(v_1) \neq \phi(v_4)$ in any optimal colouring. The argument for H is similar, though a bit more involved.

The colouring illustrated above establishes that $\chi(H) \leq 3$. Furthermore, the subgraph induced by the vertices w, x and y is isomorphic to K_3 and so we know $\chi(H) \geq 3$ as well and thus $\chi(H) = 3$. Any 3-colouring of H must assign distinct colours to the vertices u, w and x, as they are all adjacent to each other. Similar reasoning says that w, x and y must all receive distinct colours. But this means that any 3-colouring ϕ must have

$$\phi(u) = \phi(y).$$

And then, as v and y are adjacent, they must receive distinct colours. Thus it can never be true that $\phi(u) = \phi(v)$, which provides a second counterexample to the the proposition.

(9) (Bounds on $\chi(G)$). This problem is about whether one can get a bound on the chromatic number of a graph using either the the maximal degree

$$\Delta(G) = \max_{v \in V} \deg(v),$$

or the average degree

$$\text{avgdeg}(G) = \frac{\sum_{v \in V} \deg(v)}{|V|}.$$

(a) It is true that $\chi(G) \leq \Delta(G) + 1$. To see this, note that the greedy colouring algorithm will certainly be able to construct a $\Delta(G) + 1$ colouring. For when we come to choose a colour for some $v \in V$ it will have, at most, $\Delta(G)$ neighbours and so, with $\Delta(G) + 1$ colours available, there will always be an unused one that we can assign to $\phi(v)$. And the existence of a $(\Delta(G) + 1)$-colouring implies that $\chi(G) \leq \Delta(G) + 1$. This bound is sharp, as every vertex in the complete graph K_n has degree $(n - 1)$, so $\Delta(K_n) = (n - 1)$. But then $\chi(K_n) = n = \Delta(K_n) + 1$.

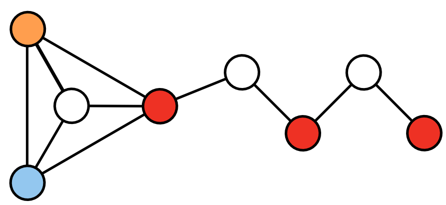


Figure 2: *A counterexample to the claim that* $\chi(G) \leq \text{avgdeg}(G) + 1$. *This graph, which consists of a subgraph isomorphic to* K_4 *(at left) and a "tail" of 4 extra vertices, has* $\chi(G) = 4$, *but* $\text{avgdeg}(G) = 5/2$.

(b) The average degree does not provide a bound on the chromatic number, as is demonstrated by the graph in Figure 2. It has $\chi(G) = 4$ because it contains a subgraph isomorphic to K_4, but has degree sequence $(1, 2, 2, 2, 3, 3, 3, 4)$ and hence average degree

$$\text{avgdeg}(G) = \frac{1 + 3 \times 2 + 3 \times 3 + 4}{8} = \frac{20}{8} = \frac{5}{2},$$

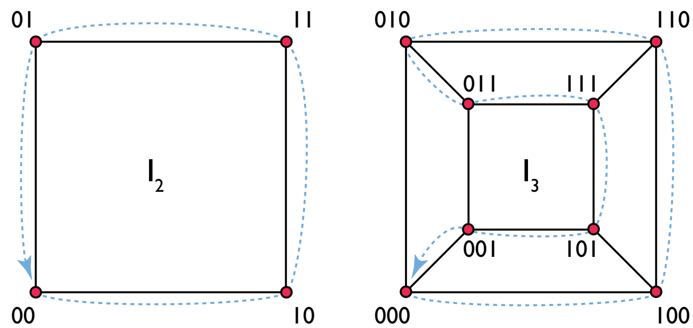
so

$$\text{avgdeg}(G) + 1 = \frac{7}{2} < \chi(G).$$

In fact, you can make similar graphs that have $\text{avgdeg}(G)$ as close to 2 as you like. The idea is to extend the tail so that it includes k vertices. Then

$$\text{avgdeg}(G) = \frac{1 + (k-1) \times 2 + 3 \times 3 + 4}{4 + k} = \frac{12 + 2k}{4 + k}.$$

(10). It is possible to construct a proof by induction along the lines of the examples in the answer to Problem (5). The key idea is that I_{d+1} is essentially two copies of I_d, glued together. The figure below shows how this works for I_2 and I_3 and suggests a way to use a cycle in I_d to construct a cycle in I_{d+1}.



Expressing the idea in words, we construct a cycle in I_3 by splitting its vertex labels into two groups: those ending in 0 and those ending in 1. Each group has a natural, one-to-one correspondence with the vertex labels in I_2 (just ignore the final digit in the labels from I_3). To get the desired cycle in I_3 we traverse the first group of vertices in the order suggested by the cycle from I_2 and then jump over to the second group and traverse that in the opposite order.

Now we'll develop a recursive algorithm that generates a sequence of vertex labels for a cycle in I_{d+1}, given one for I_d, but before we can do this we need a little notation. Let's say that $w_{j,d}$, where $0 \leq j \leq 2^d$, is the label of the j-th vertex in the cycle found in I_d. From the answer to the previous question, we could say that

$$\begin{aligned} w_{0,2} &= 00 \\ w_{1,2} &= 01 \\ w_{2,2} &= 11 \\ w_{3,2} &= 10 \\ w_{4,2} &= 00 \end{aligned}$$

As we want to build up the vertex labels recursively, we'll also need a notation to indicate concatenation of letters. We'll write $w_{j,k} \oplus 1$ to mean "append a 1 on to the end of the string $w_{j,k}$". Similarly, $w_{j,k} \oplus 0$ means "append a 0". Thus, for example,

$$\begin{aligned} w_{1,2} \oplus 0 &= 01 \oplus 0 = 010 \\ w_{1,2} \oplus 1 &= 01 \oplus 1 = 011 \end{aligned}$$

Then define the rest recursively by

$$w_{j,d+1} = \begin{cases} w_{j,d} \oplus 0 & \text{If } 0 \leq j < 2^d \\ w_{2^{d+1}-j-1,d} \oplus 1 & \text{If } 2^d \leq j < 2^{d+1} \\ w_{0,d} \oplus 0 & \text{If } j = 2^{d+1} \end{cases}$$