MATH20902: Discrete Maths, Solutions to Problem Set 1

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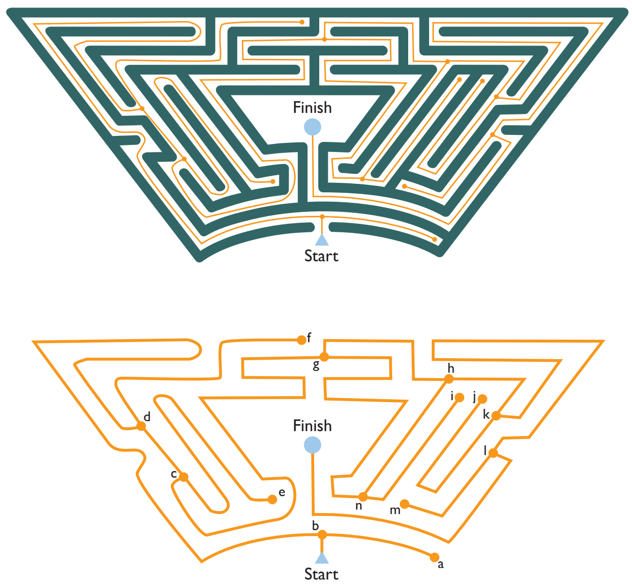
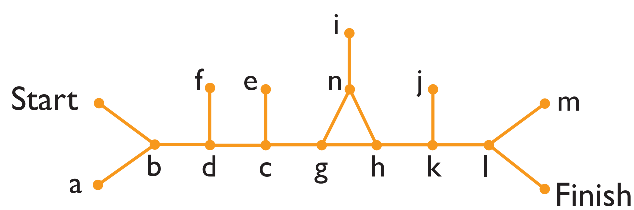
(1). The upper panel in the figure below shows the maze and the network of paths within it while the lower panel shows the network of paths as a graph with named vertices.

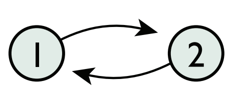


Once one has the bottom panel of the figure above, it's easy to abstract it further, to a form in which the solution is obvious:



Notice that there is not a unique path from the start vertex to the finish. The cycle that includes vertices g, h and n encloses a segment of the maze's walls that are, in a natural sense, disconnected from the rest.

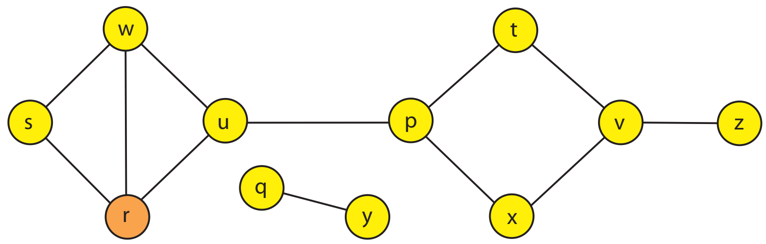
(2). The digraph



has adjacency matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(3). The graph in question is below.



(4). Given the adjacency matrix of an undirected graph, one can compute the degrees of its vertices via

$$\deg(v_j) = \sum_{k=1}^{n} A_{jk} = \sum_{k=1}^{n} A_{kj}.$$

The degrees of the vertices in H are thus

Vertex v_j	v_1	v_2	v_3	v_4	v_5	v_6	v_7
$\deg(v_j)$	5	4	4	4	4	4	5

(5). A suitable set with four elements is $\{1, 2, 3, 4\}$ and we can label the vertices of the graph T_4 with subsets of the form $\{j, k\}$. The vertex set is then

$$\{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$$

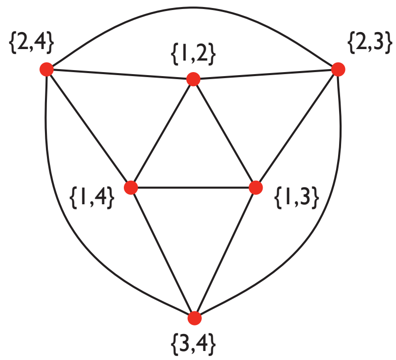
and the adjacency matrix is

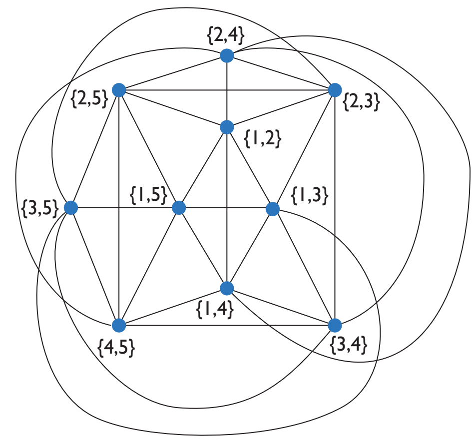
	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$
$\{1,2\}$	0	1	1	1	1	0
$\{1,3\}$	1	0	1	1	0	1
$\{1,4\}$	1	1	0	0	1	1
$\{2,3\}$	1	1	0	0	1	1
$\{2,4\}$	1	0	1	1	0	1
$\{3,4\}$	0	1	1	1	1	0

Similar considerations for T_5 lead to the adjacency matrix

	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{1,5\}$	$\{2,3\}$	$\{2,4\}$	$\{2,5\}$	$\{3,4\}$	$\{3,5\}$	$\{4,5\}$
$\{1,2\}$	0	1	1	1	1	1	1	0	0	0
$\{1,3\}$	1	0	1	1	1	0	0	1	1	0
$\{1,4\}$	1	1	0	1	0	1	0	1	0	1
$\{1,5\}$	1	1	1	0	0	0	1	0	1	1
$\{2,3\}$	1	1	0	0	0	1	1	1	1	0
$\{2,4\}$	1	0	1	0	1	0	1	1	0	1
$\{2,5\}$	1	0	0	1	1	1	0	0	1	1
$\{3,4\}$	0	1	1	0	1	1	0	0	1	1
$\{3,5\}$	0	1	0	1	1	0	1	1	0	1
$\{4,5\}$	0	0	1	1	0	1	1	1	1	0

and to the figure below, which shows T_4 on the left and T_5 on the right.





(a) The number of vertices in T_N is the same as the number of ways to choose two distinct elements from a set of N, which is

$$\binom{N}{2} = \frac{N!}{2!\,(N-2)!} = \frac{N(N-1)}{2}.$$

(b) Suppose the set and vertex labels are like those used above and consider the vertex $\{j,k\}$. There are $(N-2)$ elements in the original set (that is, numbers from $\{1,\ldots,N\}$) that are different from both j and k and each such number gives rise to a pair of vertices adjacent to $\{j,k\}$. That is, for each i such that $i \neq j$ and $i \neq k$, we have two distinct adjacent vertices: one corresponding to the subset $\{i,\, j\}$ and another corresponding to $\{i,\, k\}$. Thus there are $2(N-2) = 2N-4$ vertices adjacent to $\{j,k\}$ or, equivalently $\deg(\{j,k\}) = 2N-4$.

(c) If two vertices are adjacent, their corresponding two-element subsets of $\{1,\ldots,N\}$ share a member. Let's say that x corresponds to the subset $\{j_1,\, k\}$ while y corresponds to $\{j_2,\, k\}$, with $j_1 \neq j_2$. Now consider those elements of the original set that differ from all three of j_1, j_2 and k and define

$$U = \{1, \ldots,\, N\} \backslash \{j_1, j_2, k\}.$$

There are $(N-3)$ elements in U and each gives rise to a vertex that is adjacent to both x and y: the corresponding subsets are of the form $\{i,\, k\}$. In addition, the vertex corresponding to $\{j_1,\, j_2\}$ is adjacent to both x and y, giving a total of $(N-3)+1 = N-2$ common neighbours.

(d) If two vertices x and y are not adjacent, then their corresponding two-element subsets have no elements in common. Let's say that x corresponds to $\{j_1,\, k_1\}$ while y corresponds to $\{j_2,\, k_2\}$, where all four of the numbers j_1, j_2, k_1 and k_2 are distinct. The only way a vertex can be adjacent to both x and y is if its two-element subset shares a member with each of the subsets corresponding to x and y and there are clearly only four ways this can happen:

$$\{j_1,\, j_2\},\ \{j_1,\, k_2\},\ \{k_1, j_2\} \text{ and } \{k_1, k_2\}.$$