This notebook applies two theorems to check which members of a family of graphs *could* be planar. The first theorem says that if *G(V,E)* is planar, then either:

- *G* is a tree or
- *G* contains at least one cycle and so has a well-defined girth *g* (the girth is the length of the shortest cycle). In this case we have the bound

$$m \leq \frac{g\ (n-2)}{(g-2)}$$

where g is the girth, $m = |E|$ and $n = |V|$.

A corollary of this result that is sometimes easier to apply says that if *G* is planar, then $m \leq 3*n - 6$.

Complete graphs K_n

We begin by defining functions that count things

In[∘]:=
```
mEdges[n_] := n * (n - 1) / 2
girth[n_] := If[(mEdges[n] == (n - 1)), Null, 3];
```

Make a table whose columns are, from left to right, number of vertices, number of edges, girth, maximum number of edges allowed by the girth bound, the verdict (truth value of “This graph could be planar” in light of the theorem) from the main theorem, then number of edges allowed by the bound in the corollary and the verdict of corollary.

In[∘]:=
```
MatrixForm[
 Table[
  {n, mEdges[n],
   girth[n],
   If[(mEdges[n] == (n - 1)), Null, (girth[n] / (girth[n] - 2)) * (n - 2)],
   (n < 3) || (mEdges[n] ≤ (girth[n] / (girth[n] - 2)) * (n - 2)),
   If[(mEdges[n] == (n - 1)), Null, 3 * n - 6],
   (n < 3) || (mEdges[n] ≤ 3 * n - 6)
  },
  {n, 1, 10}
 ]
]
```

Out[∘]//MatrixForm=

1	0	Null	Null	True	Null	True
2	1	Null	Null	True	Null	True
3	3	3	3	True	3	True
4	6	3	6	True	6	True
5	10	3	9	False	9	False
6	15	3	12	False	12	False
7	21	3	15	False	15	False
8	28	3	18	False	18	False
9	36	3	21	False	21	False
10	45	3	24	False	24	False

In[∘]:=
```
Remove[mEdges, girth]
```

Complete Bipartite graphs $K_{r,s}$

The table we build here is similar to the one above, but now we parameterise the graph in terms of two integers, *r* and *s.*

In[•]:=
```
nVerts[r_, s_] := (r + s)
mEdges[r_, s_] := r * s
girth[r_, s_] := If[(r < 2) || (s < 2), Null, 4];
```

In[•]:=
```
MatrixForm[
 Flatten[
  Table[
   Table[
    {s, r,
     nVerts[r, s], mEdges[r, s], girth[r, s],
     If[(mEdges[r, s] == (nVerts[r, s] - 1)), Null,
      (girth[r, s] / (girth[r, s] - 2)) * (nVerts[r, s] - 2)],
     (mEdges[r, s] == (nVerts[r, s] - 1)) ||
      (mEdges[r, s] ≤ (girth[r, s] / (girth[r, s] - 2)) * (nVerts[r, s] - 2)),
     If[(mEdges[r, s] == (nVerts[r, s] - 1)), Null, (3 * nVerts[r, s] - 6)],
     (mEdges[r, s] == (nVerts[r, s] - 1)) || (mEdges[r, s] ≤ 3 * nVerts[r, s] - 6)
    },
    {r, 1, s}
   ],
   {s, 1, 5}
  ],
  1
 ]
]
```

Out[•]//MatrixForm=

1	1	2	1	Null	Null	True	Null	True
2	1	3	2	Null	Null	True	Null	True
2	2	4	4	4	4	True	6	True
3	1	4	3	Null	Null	True	Null	True
3	2	5	6	4	6	True	9	True
3	3	6	9	4	8	False	12	True
4	1	5	4	Null	Null	True	Null	True
4	2	6	8	4	8	True	12	True
4	3	7	12	4	10	False	15	True
4	4	8	16	4	12	False	18	True
5	1	6	5	Null	Null	True	Null	True
5	2	7	10	4	10	True	15	True
5	3	8	15	4	12	False	18	True
5	4	9	20	4	14	False	21	True
5	5	10	25	4	16	False	24	False

In[•]:=
```
Remove[nVerts, mEdges, girth]
```

Cube graphs I_d

Here the first column is the dimension *d* of the cube graph

In[∘]:=
```
nVerts[d_] := 2^d
mEdges[d_] := d * 2^(d-1) (* from the Handshaking Lemma *)
girth[d_] := If[(d < 2), Null, 4];
```

In[∘]:=
```
MatrixForm[
 Table[
  {d, nVerts[d], mEdges[d],
   girth[d],
   If[( mEdges[d] == (nVerts[d] - 1)),
    Null, (girth[d] / (girth[d] - 2)) * (nVerts[d] - 2)],
   (d < 2) || (mEdges[d] ≤ (girth[d] / (girth[d] - 2)) * (nVerts[d] - 2) ),
   If[( mEdges[d] == (nVerts[d] - 1)), Null, 3 * nVerts[d] - 6],
   (d < 2) || (mEdges[d] ≤ 3 * nVerts[d]  - 6)
  },
  {d, 1, 6}
 ]
]
```

Out[∘]//MatrixForm=

1	2	1	Null	Null	True	Null	True
2	4	4	4	4	True	6	True
3	8	12	4	12	True	18	True
4	16	32	4	28	False	42	True
5	32	80	4	60	False	90	True
6	64	192	4	124	False	186	False

In[∘]:=
```
Remove[nVerts, mEdges, girth]
```