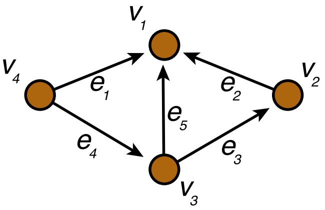
The *edge incidence matrix* M of a digraph $G(V, E)$ is constructed as follows:

- Number the vertices, so V becomes $V = \{v_1, \ldots, v_n\}$.
- Number the edges, so E becomes $E = \{e_1, \ldots, e_m\}$.
- M is an $n \times m$ matrix whose entries are given by:

$$M_{i,j} = \begin{cases} 1 & \text{if } v_i \text{ is the tip vertex of } e_j \\ -1 & \text{if } v_i \text{ is the tail vertex of } e_j \\ 0 & \text{otherwise} \end{cases}$$

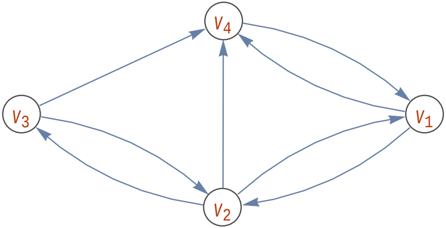


$$M = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 \\ -1 & 0 & 0 & -1 & 0 \end{bmatrix}$$

These slides are available on Blackboard and at `https://bit.ly/30n6NJW`

- Compute M for the digraph at right. Your answer will, of course, depend on the ordering you choose for the edges.
- Compute $\deg_{in}(v_j)$ and $\deg_{out}(v_j)$ for all the vertices.
- Is there a way to use sums over the entries of M to compute these degrees?

  Hint: think of sums over both $M_{i,j}$ and $|M_{i,j}|$.



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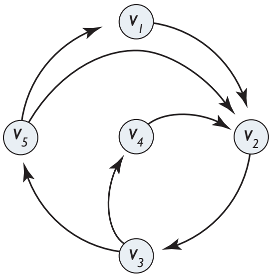

- Compute M for the digraph at left.
- Now compute the product MM^T, where M^T is the transpose of M. The product should be 5×5, with one row and column for each vertex.

Recall that the graph Laplacian of an *undirected* graph $G(V, E)$ is the matrix

$$L = D - A$$

where D is a diagonal matrix with $D_{j,j} = \deg(v_j)$ and A is the adjacency matrix of G.

- Compute the Laplacian of the graph $|G|$ formed by ignoring the directedness of the edges in the digraph above and compare it to MM^T.

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The example on the previous slide suggests an alternative approach to the construction of the Laplacian of an undirected graph:

- Convert G into a digraph by choosing an arbitrary *orientation* for each edge.
- Compute M for the resulting digraph.
- Then set $L = MM^T$.

(i) Do you think this approach will always work, no matter how you choose the orientations? If so, can you prove it?

(ii) Do you think this approach generalises straightforwardly to computing the Laplacian of a digraph? *Hint: try some small examples*.

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Afterword: why might we care about MM^T ?

The construction we've developed is—if you like this sort of thing—cool, but somewhat impractical: it's much easier to construct the Laplacian of an undirected graph using $L = D - A$ than with $L = MM^T$.

Nevertheless, MM^T is interesting becasue it is involved in Kirchoff's original proof of the Matrix-Tree theorem. He was thinking about electrical circuits in which the vertices are held at certain fixed electrical potentials (by, say, batteries) and currents flow along the edges, which represent resistors and wires. The arbitrary orientations chosen on the previous slide correspond to (also arbitrary) choices of the directions of positive currents.

Proofs of the Matrix-Tree theorem that exploit the factorisation $L = MM^T$ appear on, for example, pages 48–50 of:

John M. Harris, Jeffry L. Hirst and Michael J. Mossinghoff (2008), *Combinatorics and Graph Theory*, 2nd edition, Springer, New York.

and in Section 4.2 of:

D. Jungnickel (2013), *Graphs, Networks and Algorithms*, 3rd edition, Vol. 5 of *Algorithms and Computation in Mathematics*, Springer-Verlag, Heidelberg.

These slides are available on Blackboard and at `https://bit.ly/30n6NJW`

It's sort of amazing that, in the Kirchoff's Matrix-Tree theorem, it doesn't matter which row and and column we delete when we want to compute the number of spanning trees. That is, if $\hat{L}_j$ is the matrix formed by deleting the j-th row and column from the Laplacian L of an undirected graph, then

$$\det(\hat{L}_j) = N_T,$$

where N_T is the number of spanning trees. This turns out to be a special case of the following result, which I invite you to prove:

Theorem

If all the rows and columns of an $n \times n$ matrix A sum to zero — that is, if

$$\sum_{j=1}^{n} A_{i,j} = \sum_{j=1}^{n} A_{j,i} = 0 \qquad \textit{for } 1 \leq i \leq n,$$

— and if we define $\hat{A}_j$ to be the $(n-1) \times (n-1)$ matrix formed by deleting the j-th row and column of A, then there is some constant C such that

$$\det(\hat{A}_j) = C.$$

That is, $\det(\hat{A}_j)$ is independent of j.