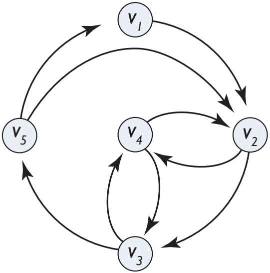
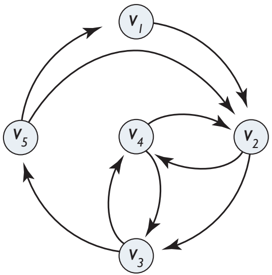


- To list all spregs with, say, distinguished vertex v_5, fill in the table below in as many ways as possible:

v	v_1	v_2	v_3	v_4
v's predecessor				

Once you've made all the tables,it's easy to draw all the spregs.

These slides are available on Blackboard and at https://bit.ly/3vkaIFz



- One can count the possibilities without actually writing them out. The predecessor of v_j has to be chosen from P_{v_j}, v_j's predecessor list, and so the number of possibilities is

$$\prod_{j \neq 5} |P_{v_j}| = \prod_{j \neq 5} \deg_{in}(v_j).$$

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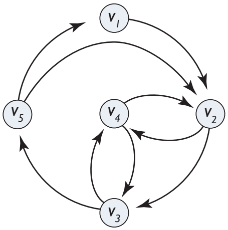
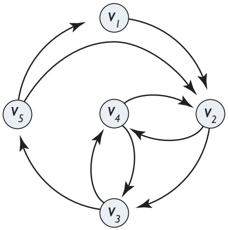


For the graph above the matrix L in Tutte's theorem is

$$L = D - A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 3 & -1 & -1 & 0 \\ 0 & 0 & 2 & -1 & -1 \\ 0 & -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 0 & 1 \end{bmatrix}.$$

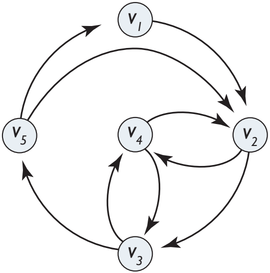


For the graph above the matrix $\hat{L}_5 \equiv \mathcal{L}$ in the proof of Tutte's theorem is

$$\mathcal{L} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 3 & -1 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}.$$

and the term in the determinant corresponding to $\sigma = \mathrm{id}$, the identity permutation, is

$$\mathrm{sgn}(\sigma) \prod_{j=1}^{4} \mathcal{L}_{j\sigma(j)} = 1 \times \prod_{j=1}^{4} \mathcal{L}_{jj} = \prod_{j=1}^{4} \deg_{in}(v_j) = 1 \times 3 \times 2 \times 2 = 12.$$



Here we'll count spregs with distinguished vertex v_5 that contain the cycle (v_3, v_4, v_3)

- Asking that a particular cycle be present fixes some of the entries in the table:

v	v_1	v_2	v_3	v_4
v's predecessor			v_4	v_3

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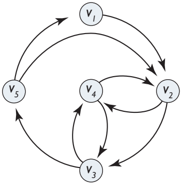


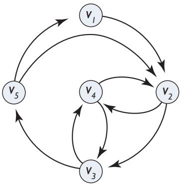
$$\mathcal{L} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 3 & -1 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}.$$

Here we'll count spregs with distinguished vertex v_5 that contain the cycle (v_3, v_4, v_3)

- Here too, one can count the possibilities without actually writing them out: the predecessor of v_j has to be chosen from P_{v_j}, the predecessor list of v_j and so the number of possibilities is

$$\prod_{j \notin \{3,4,5\}}^{5} |P_{v_j}| \;=\; \prod_{j \notin \{3,4,5\}} \deg_{in}(v_j) \;=\; 1 \times 3 = 3.$$

These slides are available on Blackboard and at https://bit.ly/3vkaIFz



$$\mathcal{L} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 3 & -1 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}.$$

The permutation that corresponds to the cycle (v_3, v_4, v_3) is the transposition $(3, 4)$ or

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix},$$

so the term in the determinant is

$$\begin{aligned} \operatorname{sgn}(\sigma) \prod_{j=1}^{4} \mathcal{L}_{j\sigma(j)} &= (-1) \times \mathcal{L}_{11} \times \mathcal{L}_{22} \times \mathcal{L}_{34} \times \mathcal{L}_{43} \\ &= (-1) \times \deg_{in}(v_1) \times \deg_{in}(v_2) \times (-1) \times (-1) \\ &= (-1) \times 1 \times 3 \\ &= -3. \end{aligned}$$