# Spanning trees in $K_3$ and $K_4$

In this week's review session we'll use Kirchoff's Matrix-Tree Theorem to count spanning trees in $K_n$.

- Draw $K_3$ and all the spaning trees it contains.
- Try to do the same thing for $K_4$: how many spanning trees can you find? *As we'll see, there are 16.*

Let's now use Kirchoff's Matrix-Tree theorem:

- Write down the graph Laplacian of $K_3$.
- Delete the first row and column of the Laplacian from above to form $\hat{L}_1$ and compute its determinant: this should match the number of spanning trees you found.
- Try the same approach for $K_4$.

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## Spanning trees in $K_n$: in general

Let's now analyse the Laplacian of $K_n$ in general:

- Show that the graph Laplacian of $K_n$ can be written as

$$L = \begin{bmatrix} (n-1) & -1 & \cdots & \cdots & -1 \\ -1 & (n-1) & -1 & \cdots & -1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & & (n-1) & -1 \\ -1 & \cdots & \cdots & -1 & (n-1) \end{bmatrix} = nI_n - M_n$$

  where $I_n$ is the $n \times n$ identity matrix and $M_n$ is an $n \times n$ matrix full of ones.
- Delete the first row and column of $L$ to form $\hat{L}_1$ and show that

$$\hat{L}_1 = nI_{n-1} - M_{n-1}.$$

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# The eigenvalues of $\hat{L}_1$

Here we find the eigenvalues of $\hat{L}_1 = nI_{n-1} - M_{n-1}$:

- Show that $u = (1, \ldots, 1) \in \mathbb{R}^{n-1}$ is an eigenvector of $\hat{L}_1$ with eigenvalue $1$.
- Show that any vector $v \in \mathbb{R}^{n-1}$ satisfying $v \cdot u = 0$ is an eigenvector of $\hat{L}_1$ with eigenvalue $n$. *Hint: the rows of $M_{n-1}$ are all equal to $u$.*
- Show that the subspace $V \subset \mathbb{R}^{n-1}$ defined by

$$V = \left\{ v \in \mathbb{R}^{n-1} \mid v \cdot u = 0 \right\}$$

  has dimension $(n-2)$. *Hint: construct a set of $n-2$ linearly independent elements of $V$.*
- Conclude that the eigenvalues of $\hat{L}_1$ are

$$1, \underbrace{n, \cdots, n}_{n-2 \text{ times}}.$$

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Finally, use the fact that, for an arbitrary $(n-1) \times (n-1)$ matrix $A$ whose eigenvalues are $\lambda_1, \ldots, \lambda_{n-1}$

$$\det(A) = \prod_{j=1}^{n-1} \lambda_j$$

to conclude that $\det(\hat{L}_1) = n^{n-2}$ and so that $K_n$ contains $n^{n-2}$ spanning trees.

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