# Colouring and Isomorphism

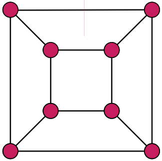
- Prove the following proposition or find a counterexample:

  *If two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are isomorphic, then $\chi(G_1) = \chi(G_2)$.*

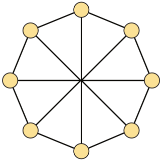
  Hint: It's helpful to start by proving

  *If $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are isomorphic and $G_1$ is $k$-colourable, then so is $G_2$.*

- Find the chromatic numbers of the graphs below, supporting your answers with rigorous arguments.



*P*



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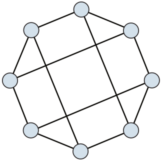

*R*

- Which, if any, of the graphs illustrated above are isomorphic?

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Suppose that $G(V, E)$ is a graph with $|V| = 5$ vertices and $|E| = 7$ edges and no loops: list all the degree sequences that $G$ could have.

(a) What is the largest number of edges that a graph on 5 vertices can have?
(b) What is the degree sequence of $K_5$?
(c) If we remove an edge from $K_5$, what degree sequences can the resulting graph have?
(d) What if we remove two edges? Three?

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Which of the following could be the degree sequence of a graph?

(a) $(1, 1, 2, 2, 2, 3, 3)$

(b) $(3, 3, 4, 4, 4, 5, 5, 5)$

(c) $(3, 3, 4, 4, 4, 5, 5)$

(d) $(1, 2, 3, 4, 5, 6, 7)$

(e) $(4, 4, 4, 5, 5, 5, 5)$

(f) $(0, 2, 2, 3, 4, 5, 6)$

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This is *not examinable*, but it's interesting and provides an easy approach to problems about degree sequences. There is an algorithm that allows one to decide whether a given sequence $\mathcal{D}$ can be the degree sequence of a graph. It depends on the following result

**Theorem (V. Havel (1955) and S.L. Hakimi (1962))**

*Given a sequence $\mathcal{D} = (d_1, d_2, \ldots, d_n)$ with $d_j \leq d_{j+1}$, there is a graph having $\mathcal{D}$ as it's degree sequence if and only if there is a graph whose degree sequence is given by $\mathcal{D}' = (d'_1, d'_2, \ldots, d'_{n-1})$ with $d'_j \leq d'_{j+1}$, where $D'$ is formed by:*

- *removing a largest element, $d_n$, from $\mathcal{D}$;*
- *subtracting 1 from each of the $d_n$ largest remaining elements;*
- *rearranging the result in ascending order (if need be).*

Think about how to use this result to decide whether a given sequence can be the degree sequence of a graph, then determine which of the following could be degree sequences:

- (2, 3, 3, 3, 3)
- (2, 2, 4, 4, 4)
- (2, 2, 2, 5, 5, 5, 5)
- (3, 3, 4, 5, 5, 5, 5)

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