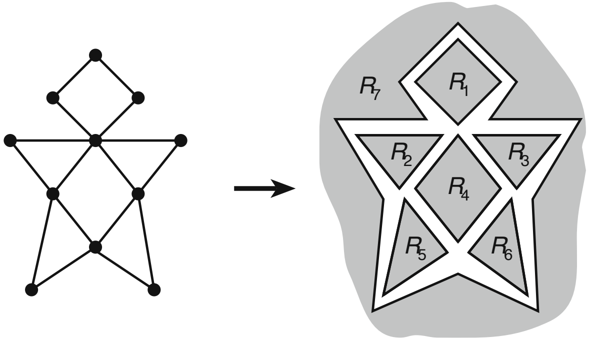
A way to think about faces

A book I often mention[1] offers this diagram to help the reader think about counting the number of faces in a planar diagram.



These slides are available on Blackboard and at https://bit.ly/3etGSXq

[1] J. M. Harris, J. L. Hirst, and M. J. Mossinghoff (2008), *Combinatorics and Graph Theory*, Undergraduate Texts in Mathematics, Springer New York, 2nd edition.

A football has 32 faces, each of which is either a regular pentagon or a regular hexagon. Because of the angles involved, exactly three faces meet at each corner. Without looking at a ball, determine how many faces of each type there are[2].

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[2] From D. A. Marcus (2008), *Graph Theory: A Problem Oriented Approach*, MAA Textbooks, MAA, 2008.

The Problem Set for this week asks us to prove:

Theorem

If $G(V, E)$ is a planar graph on $n \geq 3$ vertices and $n_d = |\{v \in V \mid \deg(v) \leq d\}|$ is the number of vertices of degree at most d, then

$$n_d \geq \frac{n(d-5)+12}{d+1}$$

That is, just by knowing that a graph is planar, we can say something about how many vertices of degree d it must have.

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Theorem (The Four Colour Theorem: Appel & Haken, 1976)

If G is a planar graph then $\chi(G) \leq 4$.

Suppose that, somehow, we could prove that a planar graph must contain at least one vertex with degree 3 or less. Then we could prove the Four Colour Theorem by induction on n using the following outline:

Base case: $|V| = 1$ and, trivially, $\chi(G) = 1 \leq 4$.

Inductive hypothesis: Assume the result is true whenever $|V| \leq n_0$.

Inductive step: Consider a planar graph $G(V, E)$ with $|V| = n_0 + 1$.

- Find a vertex v with $\deg(v) \leq 3$.
- Form $G' = G \backslash v$. It is planar and has only n_0 vertices and so, by the inductive hypothesis, has a four-colouring ϕ.
- Extend ϕ to a four-colouring of G by setting

$$\phi(v) = \text{some colour not used for one of } v\text{'s neighbours.}$$

This is guaranteed to work as $\deg(v) \leq 3$ and we have four colours available on Blackboard and.

The Four Colour Theorem remains hard

Sadly, the strongest result we can get from the inequality about n_d is:

Corollary

A planar graph has $G(V, E)$ has at least two vertices whose degree is 5 or less: that is $n_5 \geq 2$.

In fact, it's possible to construct planar graphs $G(V, E)$ in which $\deg(v) \geq 5$ for all $v \in V$.

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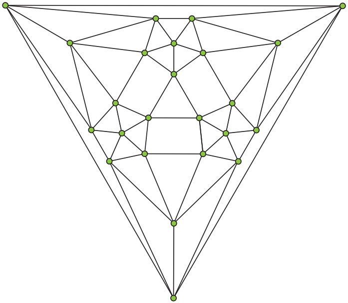
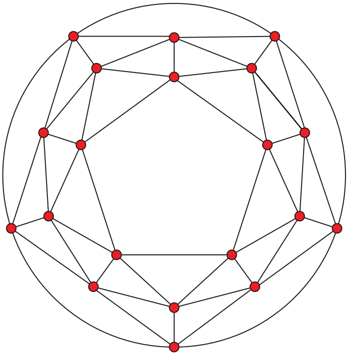
In both the graphs below, every vertex has degree five—such a graph is said to be *five-regular*—and the example at right can be extended to produce planar, five-regular graphs with an arbitrarily large number of vertices.





Both examples were drawn from J. Kanno (2005), 5-regular simple planar graphs and D-operations.

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Consider a graph $G(V, E)$ that has $|V| = 12$ vertices and $|E| = 61$ edges: that is, it is formed by removing 5 edges from K_{12}.

Answer the following questions, supporting each answer with a rigorous argument.

- Is G Eulerian?
- Is G Hamiltonian?
- What is the girth of G?
- Is G planar?

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A *regular polyhedron* has f faces, each of which is a regular (edges all the same length, internal angles all the same) s-sided polygon, and arranged so that r faces meet at every vertex. Prove that there are only five such polyhedra.

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