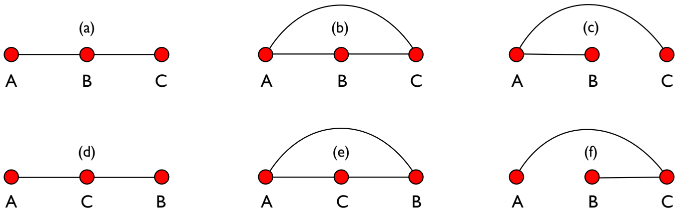
This week's first activity involves the following collection of diagrams.



Please visit `https://forms.gle/jdDd2uGztrhjDbhA8` to fill in a poll about them.

These slides are available on Blackboard and at `https://bit.ly/3p0MNGO`.

Suppose you know that an undirected graph $G(V, E)$ that doesn't contain any loops has 7 vertices and

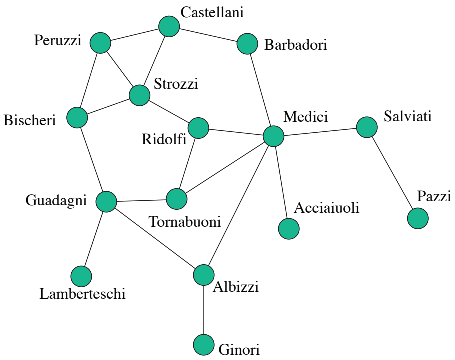
$$\min_{v \in V} \deg(v) = 3 \qquad \text{and} \qquad \max_{v \in V} \deg(v) = 5.$$

(i) Show that G must have at least 12 edges *Hint: think about the Handshaking Lemma*.

(ii) What is the largest possible number of edges that G could have?

These slides are available on Blackboard and at `https://bit.ly/3p0MNGO`.

Application: social networks



The tools we have already allow us to start to discuss a circle of applications in which graphs are used to represent connections between people—so-called *social networks*. The diagram above comes from a study[1] of marriage alliances in Renaissance Florence: the vertices represent elite families and the edges connect families whose members married.

[1] John F. Padgett and Christopher K. Ansell (1993), Robust action and the rise of the Medici 1400–1434, *American Journal of Sociology*, **98**(6):1259–1319.

Among the most natural questions to ask about a social network is "Does the graph show us who or what is important?" The answer depends on what one means by "important" and numerical scores designed to measure various kinds of importance in a network are called *measures of centrality*.

Perhaps the simplest such measure is *degree centrality* $C_D(v)$ and it is defined on the vertex set of an undirected social network $G(V, E)$ by

$$C_D(v) = \frac{\deg(v)}{n-1}$$

where $n = |V|$ is the number of vertices. The maximal possible degree is $(n-1)$, which makes it easy to see that $C_D : V \to [0, 1]$.

Family v	$C_D(v)$	Family v	$C_D(v)$	Family v	$C_D(v)$
Acciaiuoli	0.071	Ginori	0.071	Peruzzi	0.214
Albizzi	0.214	Guadagni	0.286	Ridolfi	0.214
Barbadori	0.143	Lamberteschi	0.071	Salviati	0.143
Bischeri	0.214	Medici	0.429	Strozzi	0.286
Castellani	0.214	Pazzi	0.071	Tornabuoni	0.214

Definition

Given a graph $G(V, E)$, define the set

$$\binom{V}{2} = \{(u, v) \mid u, v \in V, u \neq v\},$$

which consists of all possible non-loop edges between vertices in V. Then the graph

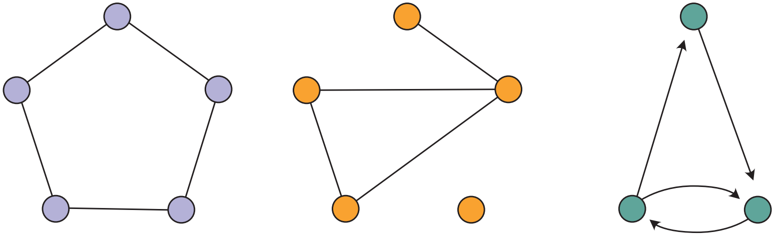
$$\overline{G}(V, \binom{V}{2} \backslash E)$$

is called the *complement* or *complementary graph* of G. Two vertices in $\overline{G}$ are adjacent if and only if they are *not* adjacent in G.

These slides are available on Blackboard and at `https://bit.ly/3pOMNGO`.

Advice on new definitions: trying examples

The first thing to do after learning a new definition is to work with a few easy examples. Find the complements of the following graphs:

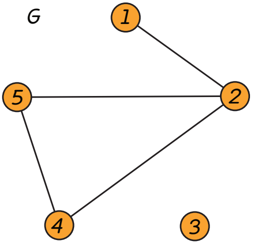


These slides are available on Blackboard and at `https://bit.ly/3pOMNGO`.

Advice on new definitions: using the new idea

The next thing to with a new definition is to see how it fits with other things you know. Fill in the table at left for the graph at right, then find a formula relating $\deg_G(v)$ and $\deg_{\overline{G}}(v)$, where the subscript indicates the graph in which we are finding the degree.

v	$\deg(v)$ in G	$\deg(v)$ in $\overline{G}$
1		
2		
3		
4		
5		



Finally, generalise your result to directed graphs, obtaining formulae relating in- and out-degrees in digraphs G and $\overline{G}$.

These slides are available on Blackboard and at `https://bit.ly/3p0MNGO`.

Definition

A graph $G(V, E)$ is called *k-regular* if $\deg(v) = k$ for all $v \in V$.

Definition

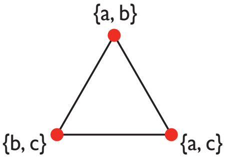
A graph $G(V, E)$ is called *strongly regular* if it is k-regular and, further, there are two constants a and b such that:

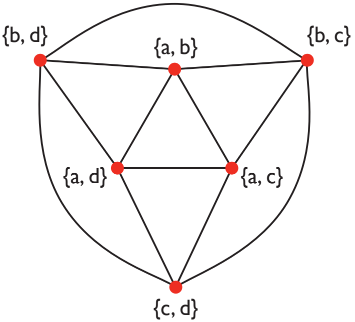
$$|A_u \cap A_v| = \begin{cases} a & \text{if } u \text{ and } v \text{ are adjacent} \\ b & \text{if } u \text{ and } v \text{ are not adjacent} \end{cases}$$

where A_u and A_v are, respectively, the adjacency lists of the vertices u and v.

These slides are available on Blackboard and at `https://bit.ly/3pOMNGO`.

Example: The *triangular graphs* T_N defined in this week's problem set are strongly regular with $k = 2N - 4,\ a = N - 2$ and $b = 4$. T_3 and T_4 are shown below.





Problem: Show that if a graph $G(V, E)$ is strongly regular with parameters k, a and b, then its complement $\overline{G}$ is also strongly regular and obtain formulae for its parameters $k',\ a'$ and b' in terms of $n = |V|,\ k,\ a$ and b.

These slides are available on Blackboard and at `https://bit.ly/3pOMNGO`.