MATH20902: Discrete Maths, Problem Set 9

This problem set is related to the material on planar graphs. Some questions refer to Dieter Jungnickel's book, *Graphs, Networks and Algorithms*, which is available online from within the University's network at `https://bit.ly/Jungnickel4`.

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(1) (Euler's theorem and graphs from regular polyhedra)**.**
We have often thought of the cube graph I_3 as being produced by squashing a wire-frame model of a cube flat. Draw a planar diagram of I_3 and find n, the number of vertices; f the number of faces and m, the number of edges and use them to verify Euler's theorem

$$n - m + f = 2.$$

Now do the same thing for the tetrahedron, the octahedron, the dodecahedron and the icosahedron.

(2) (After Jungnickel's exercise 1.5.13)**.** What is the minimum number of edges that one has to remove from K_n to get a planar graph? For each n, construct a planar graph having as many edges as possible.

(3) (Edges and bridges)**.**
An edge e in a connected graph G is a *bridge* if the graph $G \backslash e$ formed deleting e from G is not connected.

(a) Sketch two planar graphs, one that includes a bridge and one that does not.

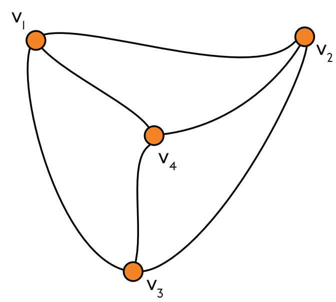
(b) Prove that if an edge is not a bridge, then it is part of at least one cycle.

(4) (After Jungnickel's exercise 1.5.14)**.** Let G be a planar graph on $n \geq 3$ vertices and denote the number of vertices of degree at most d by n_d. Prove

$$n_d \geq \frac{n(d-5)+12}{d+1}$$

(5) (Direct proofs that K_5 and $K_{3,3}$ aren't planar)**.** We will prove that K_5 isn't planar using a hard-won inequality that bounds the number edges in a planar graph in terms of the graph's girth and number of vertices, but it is also possible to prove that K_5 is non-planar more directly, by contradiction.

Take the vertex set to be $\{v_1, v_2, v_3, v_4, v_5\}$ and assume that we have a planar diagram. Consider the cycle $C \equiv (v_1, v_2, v_3, v_1)$: as the diagram is planar, the points and arcs corresponding to this cycle must form a Jordan curve. Without loss of generality, we can assume that v_4 is in the interior of this curve. And from this assumption it follows that, except for their endpoints, the arcs corresponding to the edges (v_1, v_4), (v_2, v_4) and (v_3, v_4) must also lie in the interior of C: see the figure below.



Now define $C_1 \equiv (v_2, v_3, v_4, v_2)$, $C_2 \equiv (v_1, v_3, v_4, v_1)$ and $C_3 \equiv (v_1, v_2, v_4, v_1)$ and note that v_j is in the exterior of C_j for $j = 1$, 2 and 3. Then proceed as follows:

(a) Prove that the planarity of the diagram implies that v_5 lies in the intersection of the exteriors of the C_j.

(b) Show that this implies that the arc (v_4, v_5) must cross C, which contradicts planarity.

(c) Invent a proof in the same style to show that $K_{3,3}$ can't have a planar diagram.