MATH20902: Discrete Maths, Problem Set 8

This problem set is related to the material on distance in graphs. Some questions refer to Dieter Jungnickel's book, *Graphs, Networks and Algorithms*, which is available online from within the University's network at `https://bit.ly/Jungnickel4`.

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(1). Make a drawing of a graph whose vertex set is

$$V = \{p,\ q,\ r,\ s,\ t,\ u,\ v,\ w,\ x,\ y,\ z\}$$

and whose adjacency lists are

$$\begin{array}{lll} A_p = \{t, u, x\} & A_q = \{y\} & A_r = \{s, u, w\} \\ A_s = \{r, w\} & A_t = \{p, v\} & A_u = \{p, r, w\} \\ A_v = \{t, x, z\} & A_w = \{u, r, s\} & A_x = \{p, v\} \\ A_y = \{q\} & A_z = \{v\} & \end{array}$$

Use BFS to find the distances between node r and all the others. Describe two paths of shortest length from r to z.

(2) (Basic tropical arithmetic)**.** Compute the following quantities

(a) $1 \oplus 3$	(d) $1 \otimes 3$	(g) $0 \oplus \infty$
(b) $1 \oplus -1$	(e) $1 \otimes -1$	(h) $0 \otimes \infty$
(c) $0 \oplus 5$	(f) $0 \otimes 5$	(i) $3 \otimes (3 \oplus 5)$

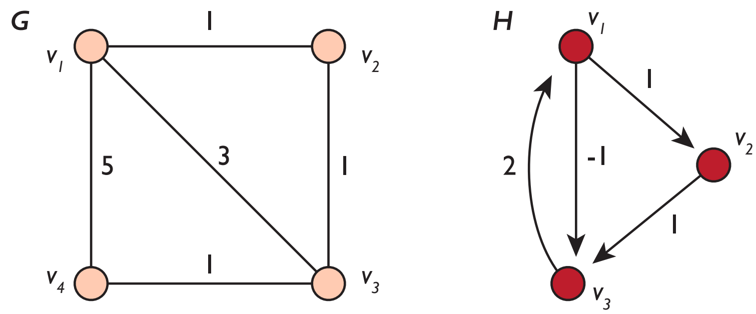
(3) (Tropical matrix products)**.** Compute the tropical matrix product

$$\begin{bmatrix} 1 & 2 \\ 0 & \infty \end{bmatrix} \otimes \begin{bmatrix} \infty & -1 \\ 1 & \infty \end{bmatrix}$$

and the element-by-element tropical matrix sum

$$\begin{bmatrix} 1 & 2 \\ 0 & \infty \end{bmatrix} \oplus \begin{bmatrix} \infty & -1 \\ 1 & \infty \end{bmatrix}.$$

(4) (Distance in graphs)**.** In the graphs below edge weights appear next to the corresponding edges



For each of the graphs above:

(a) By inspection, construct a matrix D whose entries are

$$D_{j,k} = \begin{cases} d(v_j, v_k) & \text{if } v_k \text{ is reachable from } v_j \\ \infty & \text{otherwise} \end{cases}$$

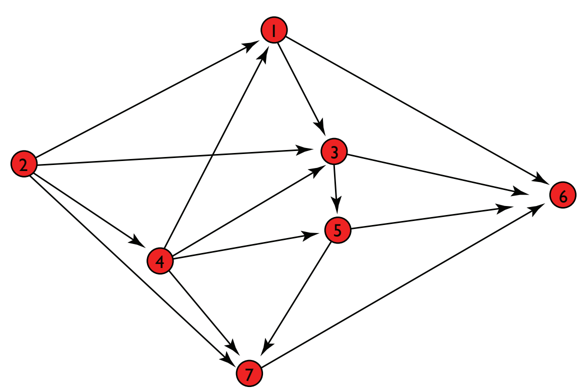
Here $d(v_j, v_k)$ is the weight of a minimal-weight walk from v_j to v_k.

(b) Construct the weight matrix W whose entires are

$$W_{j,k} = \begin{cases} 0 & \text{if } j = k \\ w(v_j, v_k) & \text{if } j \neq k \text{ and } (v_j, v_k) \in E \\ \infty & \text{otherwise} \end{cases}$$

(c) Finally, compute the tropical matrix power $W^{\otimes(n-1)}$ where $n = |V|$ is the number of vertices in the graph. The results should agree with your answers to part (a).

(5) (After Jungnickel's Exercise 2.6.9)**.**
Find a topological sorting of the graph below. That is, find a way to renumber the vertices so that if there is a directed edge (i, j) running from vertex i to vertex j, then $i < j$.



(6) (A scheduling problem)**.** Consider the following table, which describes tasks required to build a timber-framed house.

Task		*Days Required*	*Prerequisites*
A	Site preparation	4	None
B	Foundation	6	A
C	Drains & services	3	A
D	Frame walls	10	B
E	Roof	5	D
F	Windows	2	E
G	Plumbing	4	C & E
H	Electrical work	3	E
I	Insulation	2	G & H
J	Shell	6	F
K	Put up plasterboard	5	I & J
L	Cleanup and paint	3	K
M	Flooring and trim	4	L
N	Final inspection	10	I

(a) Draw a directed graph that shows the way in which the various tasks depend on each other.

(b) If you have not done so already, add a start vertex s and a finish vertex z to your graph and label all edges with the appropriate weights.

(c) Find the critical path or paths: what is the shortest time in which such a house could be built?

(d) Are there any tasks whose late completion would not affect a contractor's ability to finish the project on time? If so, how late can they be?

(7) (After Jungnickel's Example 3.1.2)**.**
Consider a road network in which junctions are the vertices and stretches of road are the edges that connect them. Say that there is a fixed probability $p(e)$ of being caught and ticketed while speeding along edge e and assume (somewhat unrealistically) that the probabilities of getting caught on distinct edges are independent, so that the probability of the very worst possible case—in which you getting caught on every single stage of your journey from vertex a to b—is given by

$$\prod_{j=1}^{l} p(e_j).$$

where I've assumed that your trip runs over the sequence of edges $P = (e_1, e_2, \ldots, e_l)$ and, as usually, $e_j = (u_j, v_j)$ and $u_1 = a$, $v_l = b$ and $u_{j+1} = v_j$ for $1 \leq j < l$.

Formulate the problem of finding the safest (in the sense of not getting caught) way to speed from a to b as a shortest-path problem in a suitable graph.

(8) (Interesting, but not examinable: Tropical polynomials)**.**
Plot the tropical polynomial

$$p(x) = 1 \otimes x^{\otimes 2} \ \oplus \ 2 \otimes x \ \oplus \ 5$$

Here $x^{\otimes n}$ is the n-th tropical power of x

$$x^{\otimes n} \ = \ \underbrace{x \otimes \cdots \otimes x}_{n \text{ times}}$$

(9) (Harder: after Jungnickel's Exercise 3.1.3)**.**
Consider the problem of an itinerant pedlar in the age of horse travel. When deciding what things to try to sell, he can choose from a set of n items, each of which has a *weight* (in, say, grams) a_j and a *value* c_j and, to keep things simple, assume that the a_j and c_j are all positive integers. He would like to find the most valuable collection of objects such that the sum of their weights does not exceed b, the upper limit on the weight that his faithful horse can carry. Reduce this problem to that of finding the longest path in a suitable directed graph.