MATH20902: Discrete Maths, Problem Set 7

Some of the problems below refer to Dieter Jungnickel's book, *Graphs, Networks and Algorithms*, an online copy of which is available from within the university at `https://bit.ly/Jungnickel4`, others mention Daniel Marcus's *Graph Theory: A Problem Oriented Approach* which is also available from within the university's network.

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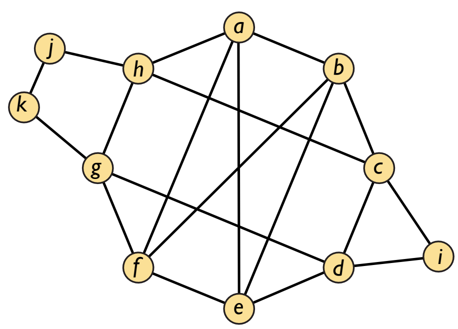
(1) (after Marcus's F5). How many bridges would have to have been built in Königsberg before it became possible to make a walking tour that crossed each bridge exactly once and finished where it started?

(2) (after Marcus's F9). Construct an Eulerian tour for the graph below as follows:

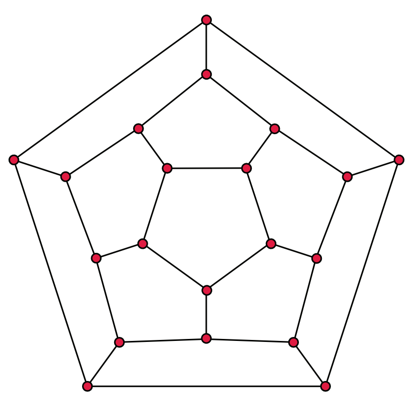


(a) Remove the edges of the cycle specified by the vertex sequence $(a, b, c, d, e, f, g, h, a)$.

(b) In each connected component of the resulting graph, find an Eulerian tour.

(c) Combine these tours with the cycle from part (a) to get an Eulerian tour for the whole graph.

(3). Find a Hamiltonian cycle in the graph pictured below.



This problem is related to the Icosian Game, a board game invented by Hamilton in the 1850's and marketed throughout Europe.

(4). The following questions are about the cube graphs I_d.

(a) For which values of d is the cube graph I_d different from its closure $[I_d]$?

(b) For which values of d is the cube graph Eulerian?

(c) For which values of d is the cube graph Hamiltonian?

(5) (after Marcus's F8)**.** Prove by contradiction that if a multigraph $G(V, E)$ has $|E| > 0$ and every vertex has even degree, then G must contain at least one cycle. *Hint: Suppose it doesn't. Show that it must then be a forest, one of whose components contains two or more vertices and thus two leaves.*

(6) (after Jungnickel's Exercise 1.3.3)**.**
Let G be a connected multigraph with exactly $2k$ vertices of odd degree (and $k \neq 0$). Show that the edge set of G can be partitioned into k trails.

(7). Consider a graph $G(V, E)$ and prove that if $\deg_G(v) < 2$, then v has the same degree in both G and $[G]$. That is, the closure construction never adds an edge to a vertex v with degree less than two.

(8) (after Marcus's G42)**.** Prove that every graph $G(V, E)$ that has degree sequence $(2, 3, 3, 4, 4, 5, 5)$ must be Hamiltonian as follows:

(a) Split the vertex set into two disjoint pieces, $\mathcal{L}$ (for "low") and $\mathcal{H}$ (for "high") defined as follows

$$\mathcal{L} = \{u \in V \mid \deg(u) \leq 3\} \qquad \text{and} \qquad \mathcal{H} = \{v \in V \mid \deg(v) \geq 4\}$$

and then argue that if $u \in \mathcal{L}$ then there exist at least

$$|\mathcal{H}| - \deg(u) = 4 - \deg(u)$$

vertices $v \in \mathcal{H}$ that are not adjacent to u.

(b) Prove, starting from the observation in part 1, that every vertex in $\mathcal{L}$ has higher degree in $[G]$ than it does in G.

(c) Use Ore's Theorem and the Bondy-Chvátal Theorem to prove that G must Hamiltonian.

(9) (after Jungnickel's Exercise 1.4.5)**.**
Find the minimal number of edges needed to make a graph G on six vertices whose closure $[G]$ is the complete graph K_6.

Part of an old exam problem

The exam question mentioned below also included some parts about planar graphs, but we will not cover that topic until later in the term, so I've removed them. I won't provide solutions to these problems, but we can discuss it in the tutorials and I will provide informal feedback on written answers.

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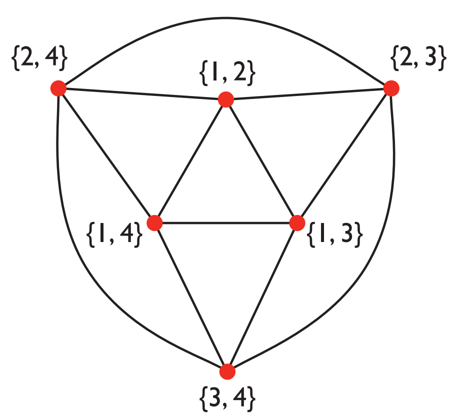
(10) (2018 B6). (a) Given a connected graph $G(V, E)$ explain what is meant by the following statements:

- G is *Hamiltonian*
- G has an *Eulerian tour*

Recall that for $N \geq 2$, the *triangular graph* T_N has vertices corresponding to the two-element subsets of an underlying set with N elements. The edge set of T_N includes an edge between two vertices if and only if the corresponding two-element subsets have a non-empty intersection. Thus if the underlying four-element set is $\{1, 2, 3, 4\}$, then T_4 is illustrated below.



(b) Prove that each vertex of T_N has degree $2N - 4$.

(c) Answer the following questions which concern T_6, and note that T_6 is *not* the graph illustrated above. Support your answers with rigorous arguments: you may use any theorem from the lectures or problem sets without providing a proof.

- Is T_6 Eulerian?
- Is T_6 Hamiltonian?