MATH20902: Discrete Maths, Problem Set 6

These problems are concerned with ideas used in the proof of Tutte's Matrix Tree Theorem.

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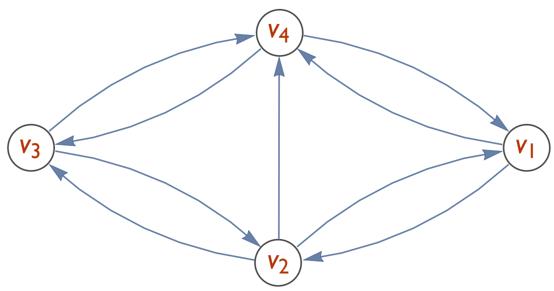
(1) (**Counting Spregs**).
Recall that, for a digraph $G(V, E)$, a ***single-predecessor subgraph (spreg) with distinguished vertex*** v is a subgraph $T(V, E')$ with the properties that

$$\deg_{in}(v) = 0 \quad \text{and} \quad \deg_{in}(u) = 1 \;\; \forall u \neq v \in V.$$



(a) How many spregs with distinguished vertex v_2 and vertex set $\{v_1, \ldots, v_4\}$ are there in the graph above?

(b) Sketch all the spregs from part (a), indicating which are spanning arborescences.

(c) Now consider spregs with distinguished vertex v_1 and answer the following:

- How many of them are there?
- How many of them contain the cycle (v_2, v_4, v_3, v_2)?
- Which element of S_3 corresponds to the term in $\det(\hat{L}_1)$ that counts spregs containing this cycle?

(2) (**A useful lemma**).
Prove the following:

Lemma (Characterising Spregs). *Prove that if $T(V, E)$ is spreg with distinguished vertex v then exactly one of the following statements is true:*

- *T is a spanning arborescence rooted at v;*
- *T contains a cycle.*

Old exam problem

I will not provide a written solution to this problem, but we can discuss it in the tutorial and I will provide informal feedback on written answers.

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(3) (Problem A2 from 2017's exam).

(a) Say what is meant by the following terms:

- a *spanning arborescence rooted at* v in a directed graph $G(V, E)$;
- a *single predecessor graph (spreg) with distinguished vertex* v in a directed graph $G(V, E)$.

(b) How many spanning arborescences rooted at v_5 are contained in the graph below?

(c) How many spregs with distinguished vertex v_5 appear the graph below?