MATH20902: Discrete Maths, Problem Set 5

These problems are all concerned with the Matrix Tree Theorems

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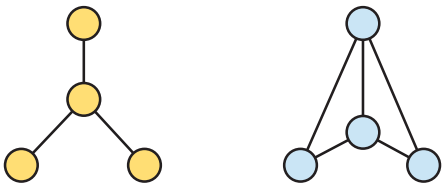
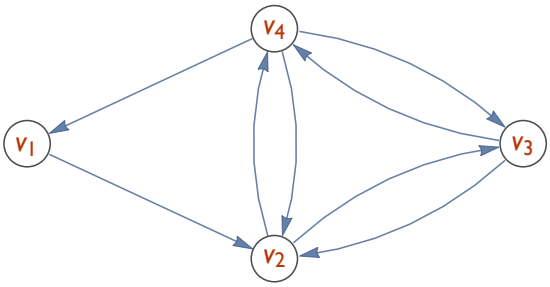
(1) (Kirchoff's Matrix Tree Theorem).
Construct the graph Laplacians for the two graphs illustrated below and use Kirchoff's Matrix-Tree Theorem to count the number of spanning trees in each, then sketch the spanning trees.



(2) (Tutte's Matrix Tree Theorem).
Construct the graph Laplacian for the graph illustrated below and the use Tutte's Matrix-Tree Theorem to count the number of spanning arborescences rooted at each of the four vertices.



(3) (Permutations and Cycles).
Choose four arbitrary elements σ from the permutation group S_6 and, for each, compute $\mathrm{fix}(\sigma)$, $\mathrm{sgn}(\sigma)$ and the cycle decomposition.

(4) (Inclusion/Exclusion).
The method of counting primes illustrated below becomes impractical for large upper limits, but the example here is tractable.

(a) Prove that if a positive integer n is composite (that is, not a prime) then n has a prime factor p satisfying $p \leq \sqrt{n}$.

(b) Use the Principle of Inclusion/Exclusion to count the number integers n in the range $2 \leq n \leq 120$ that are multiples of 2, 3, 5 or 7.

(c) How many prime numbers p lie in the range $2 \leq p \leq 120$?