MATH20902: Discrete Maths, Problem Set 4

The problems below are arranged, roughly, in increasing order of difficulty.

- They have to do with trees.
- Some of the problems below refer to Dieter Jungnickel's book, *Graphs, Networks and Algorithms* while others come from Marcus's *Graph Theory: A Problem Oriented Approach* or Bondy and Murty's *Graph Theory*. Online versions of the first two of these books are available from within the university's network at `https://bit.ly/Jungnickel4` and `https://bit.ly/3aXcV05`, respectively.

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(1) (Part of Question B4 on 2019's exam).
Suppose the internal nodes of a tree T have degrees $\{4, 3, 3, 3, 2, 2, 2\}$. Find all the degree sequences that T can have and, for each one, draw an example of a tree with that degree sequence.

(2) (after Jungnickel's Exercise 4.1.2).
A connected graph is said to be *unicyclic* if it contains exactly one cycle. Show that the following statements are equivalent:

(i) G is unicyclic.

(ii) $G \backslash e$ (the graph formed by deleting the edge e from G) is a tree for a suitable edge e.

(iii) G is connected and has the same number of edges as vertices.

(3) (after Jungnickel's Lemma 4.1.1).

Let $G(V, E)$ be a graph. Prove that the following three conditions are equivalent:

(i) G is a tree.

(ii) G does not contain any cycles, but adding any further edge creates a cycle.

(iii) Any two vertices in G are connected by a unique path.

(4) (Exercise 4.1.1 in Bondy & Murty's *Graph Theory*).
Show that every tree with maximum degree k has at least k leaves. Which such trees have *exactly* k leaves?

(5) (Exercise 4.1.7 in Bondy & Murty's *Graph Theory*).
Show that a non-decreasing sequence $(d_1, d_2, \ldots, d_n)$ of positive integers can be the degree sequence of a tree if and only if

$$\sum_{j=1}^{n} d_j = 2(n-1).$$

(6) (Adapted from Alan Gibbons (1985), *Algorithmic Graph Theory*).
A *rooted binary tree* is a rooted tree in which every node, other than the root and the leaves, has two daughters and hence degree three.

(a) Draw three examples of a rooted binary tree with 4 leaf nodes (that is, 4 leaves in addition to the root). Which, if any, of your examples are isomorphic?

(b) Prove that a rooted binary tree with $n \geq 2$ leaf nodes must have an even number of vertices.

(c) How many nodes does a binary tree with n leaves have? Prove your result.

(d) How many edges does a binary tree with n leaves have? Prove your result.

(e) **[Harder]** If the leaves are distinguishable, (say, each has a distinct label), how many distinct binary trees with n leaves are there?