MATH20902: Discrete Maths, Problem Set 3

The problems below are arranged, roughly, in increasing order of difficulty.

- They have to do with asymptotic bounds on efficiency and with connectedness.
- Some of the problems below refer to Dieter Jungnickel's book, *Graphs, Networks and Algorithms* while others come from Harris, Hirst and Mossinghoff's, *Graph Theory and Combinatorics* or Marcus's *Graph Theory: A Problem Oriented Approach.* Online versions of all these books are available from within the university's network at `https://bit.ly/Jungnickel4`, `https://bit.ly/1cp43xp` and `https://bit.ly/3aXcV05`, respectively.

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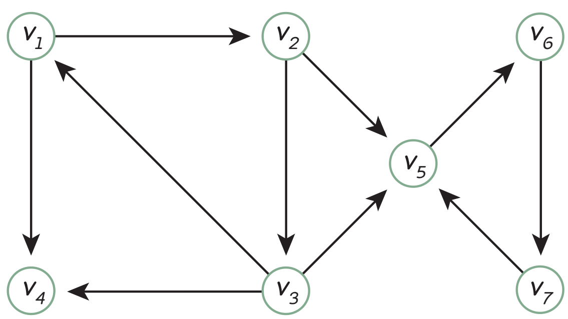
(1) (Adapted from Harris *et al.*, Exercise 7 in Section 1.1.2)**.** Draw a connected graph having at most 10 vertices that has at least one cycle of each length from 5 through 9, but no cycles of any other length.

(2) (Part of question B5 from 2018's exam)**.** Prove that strong connectedness of vertices is an equivalence relation on the vertex set of a directed graph and find the strongly connected components of the directed graph G below.



(3) (after Marcus's A39)**.** Suppose that an edge is added to a graph joining two existing vertices that were not previously adjacent.

(a) What can happen to the to the number of connected components in G? Draw examples to illustrate all possibilities.

(b) Answer the question from part (a) if, instead, we *remove* an edge from G, but leave its endpoints in place.

(4) (Arithmetic with asymptotic bounds)**.** Given two functions f_1 and f_2, each with associated asymptotic bounds $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$:

(a) Show $f_1(n)f_2(n) = O(g_1(n)g_2(n))$.

(b) If, for all sufficiently large n, $g_2(n) > g_1(n)$, show $(f_1(n) + f_2(n)) = O(g_2(n))$.

(5) (Jungnickel's Exercise 1.2.2)**.** Let G be a graph with n vertices and assume that each vertex has degree at least $(n-1)/2$. Show that G must be connected.

(6). Ask yourself whether all three of the properties of an equivalence relation really are necessary. In particular, find a counterexample to the statement: "Every symmetric, transitive relation is reflexive." That is, figure out what is wrong with the proof "$x \sim y$ and $y \sim x$ (by symmetry) implies $x \sim x$ (by transitivity)".

(7) (A polynomial in n of degree k is $\Theta(n^k)$)**.** Show that, in general, a polynomial $f : \mathbb{N} \to \mathbb{R}^+$ defined by

$$f(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_0$$

satisfies both $f(n) = O(n^k)$ and $f(n) = \Omega(n^k)$, so $f(n) = \Theta(n^k)$.