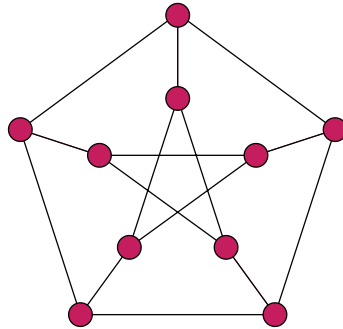


MATH20902: Discrete Maths, Problem Set 10

This problem set is related to the material on planar graphs. Some questions refer to Dieter Jungnickel's book, *Graphs, Networks and Algorithms*, which is available online from within the University's network at <https://bit.ly/Jungnickel4>.

(1) (After Jungnickel's exercise 1.5.10). The *Petersen graph* is shown below:



Prove that it is not a planar graph in the following two ways:

- (a) using the bound on m proved in lecture;
- (b) using Kuratowski's theorem.

(2). Draw three non-isomorphic graphs that are all homeomorphic to $K_{3,3}$ and, for each, compute the difference $|E| - |V|$. That is, find the difference between the number of edges and the number of vertices.

(3) (After Jungnickel's exercise 1.5.7). Let $H(V, E)$ be a graph with vertex set V and edge set E . Similarly, let $H'(V', E')$ be a second graph with vertex set V' and edge set E' . Prove that if H and H' are homeomorphic, then

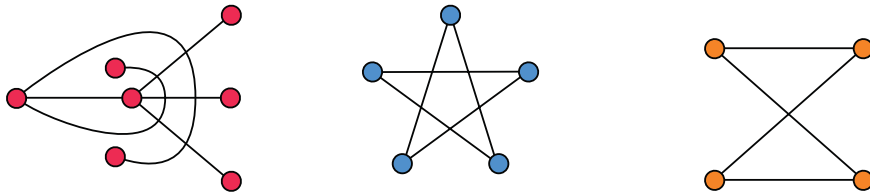
$$|E| - |V| = |E'| - |V'|.$$

(4) (A problem related to a conjecture by John Conway). The diagrams studied in this problem are a kind of opposite to planar diagrams. In a planar diagram the curves representing edges aren't allowed to cross at all, but in the diagrams discussed here they *must* do so.

A diagram for a graph is said to be a *thrackle*¹ or a *thrackle embedding* if:

- the curves representing edges do not cross themselves, but
- every pair of distinct curves cross exactly once, either at their endpoints (that is, both members of the pair are incident on a common vertex) or somewhere in their interiors.

The diagrams below illustrate two thrackles (left and center) and one diagram that isn't a thrackle (at right).



Conway conjectured that if a graph has a thrackle embedding then $m \leq n$. We'll consider some cases where either $m = n$ or $m = n - 1$.

Prove that:

- the cycle graph C_3 has a thrackle embedding;
- all cycle graphs C_k for which $k \geq 3$ is an odd number have a thrackle embedding;
- the cycle graph C_4 does not have a thrackle embedding.

¹Apparently "thrackle" is a somewhat obscure dialect word for a tangle, such as a tangled-up fishing line. Dan Archdeacon at the University of Vermont reports (<https://bit.ly/thrackle>) that Conway has said:

When I was a teenager, on holiday with my parents in Scotland, we once stopped to ask directions of a man who was fishing by the side of a lake. He happened to mention that his line was thrackled. I'd previously called this kind of drawing a tangle, but since I'd just found a knot-theoretical use for that term, I changed this to thrackle. Several people have told me that they've searched in vain for this word in dialect dictionaries, but since I quizzed the fisherman about it, I'm sure I didn't mishear it; he really did use it.