MATH20902: Discrete Maths, Problem Set 2

The problems below are arranged, roughly, in increasing order of difficulty.

- These problems have to do with subgraphs, graph isomorphism and colouring.
- Some of the problems below refer to Dieter Jungnickel's book, *Graphs, Networks and Algorithms* while others come from Harris, Hirst and Mossinghoff's, *Graph Theory and Combinatorics.* PDF versions of both these books are available from within the university's network at https://bit.ly/Jungnickel4 and https://bit.ly/1cp43xp, respectively.

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(1) (Chromatic numbers for famous graphs)**.** Find expressions for the chromatic number of each member of the following families of graphs, taking care to justify your answers rigorously:

(a) The path graphs P_n

(b) The cycle graphs C_n

(c) The cube graphs I_d

(2) (Avoiding clashes)**.** The keeper of The San Diego Zoo has been thinking of rearranging the exhibits so that animals are shown living together in their natural habitats. Unfortunately, she cannot combine animals arbitrarily as some of them prey on others. The table below summarizes these restrictions, indicating with a dot pairs of animals that should not be housed together. Determine the smallest number of enclosures needed to house all the animals safely.

	a	b	c	d	e	f	g	h	i	j
a		•			•					•
b	•			•			•			
c								•		•
d		•				•				
e	•								•	
f				•						•
g		•								
h			•						•	
i					•			•		•
j	•		•			•			•	

(3) (Adapted from Harris *et al.*, Exercise 3 in Section 1.1.3)**.** Is K_4 a subgraph of $K_{4,4}$? If so, then illustrate this and if not, then explain why not.

(4) (Adapted from Harris *et al.*, Exercise 4 in Section 1.1.3)**.** Is P_4 an *induced* subgraph of $K_{4,4}$? If not, explain why not, but if so, then illustrate this by drawing $K_{4,4}$ and labeling its vertices, then listing a subset of the vertices whose induced subgraph is isomorphic to P_4.

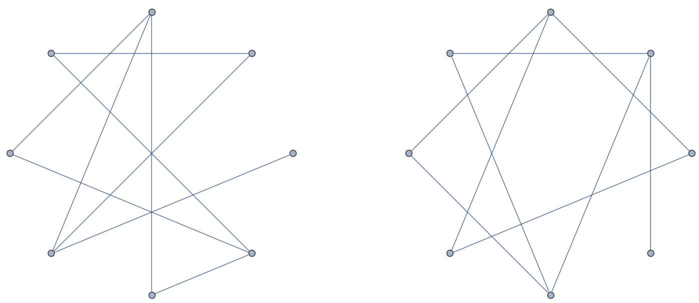
(5). Show that:

(a) the cube graph I_2 contains a subgraph isomorphic to the cycle graph C_4

(b) the cube graph I_3 contains a subgraph isomorphic to the cycle graph C_8

(c) the cube graph I_4 contains a subgraph isomorphic to the cycle graph C_{16}

(6) (Adapted from Harris *et al.*, Exercise 9 in Section 1.1.3)**.** Prove that the two graphs below are not isomorphic.



Hint: it may prove helpful to label the vertices, write out the adjacency lists and then redraw the graphs.

(7) (Adapted from Harris *et al.*, Exercise 2 in Section 1.1.2)**.** Prove that for any graph $G(V, E)$ with $|V| \geq 2$, the degree sequence has at least one repeated entry.

(8) (Colouring non-adjacent vertices)**.** Prove the following or find a counterexample:

Proposition. *If $G(V, E)$ is a graph and $k = \chi(G)$ is its chromatic number then, for any two non-adjacent vertices u, $v \in V$, there exists some k-colouring ϕ such that $\phi(u) = \phi(v)$.*

Informally, this intuitively natural proposition says that if u and v aren't adjacent, then there exists some optimal colouring that assigns them the same colour: the question asks you to find out whether this really is true.

(9) (Bounding the chromatic number)**.** The second part of this question comes from Harris, Hirst and Mossinghoff's, *Graph Theory and Combinatorics.*

(a) Say that a graph $G(V, E)$ has maximal degree $\Delta(G)$. That is, define

$$\Delta(G) = \max_{v \in V} \deg(v).$$

Prove $\chi(G) \leq \Delta(G) + 1$ or find a counterexample.

(b) For a graph $G(V, E)$, define the average degree avgdeg(G) as follows:

$$\text{avgdeg}(G) = \frac{\sum_{v \in V} \deg(v)}{|V|}.$$

Prove $\chi(G) \leq \text{avgdeg}(G) + 1$ or find a counterexample.

(10). Prove by induction on d that C_{2^d} is a subgraph of I_d for all $d \geq 2$.