Lecture 13

Tropical Arithmetic and Shortest Paths

This lecture introduces *tropical arithmetic*[1] and explains how to use it to calculate the lengths of all the shortest paths in a graph.

Reading:
The material here is not discussed in any of the main references for the course. The lecture is meant to be self-contained, but if you find yourself intrigued by tropical mathematics, you might want to look at a recent introductory article

> David Speyer and Bernd Sturmfels (2004), Tropical mathematics. Lecture notes from a Clay Mathematics Institute Senior Scholar Lecture, Park City, Utah, 22 July 2004. Available as preprint 0408099 from the arXiv preprint repository.

Very keen, mathematically sophisticated readers might also enjoy

> Diane Maclagan and Bernd Sturmfels (2015), *Introduction to Tropical Geometry*, Vol. 161 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI. ISBN: 978-0-8218-5198-2

while those interested in applications might prefer

> B. Heidergott, G. Olsder, and J. van der Woude (2006), *Max Plus at Work*, Vol. 13 of the *Princeton Series in Applied Mathematics*, Princeton Univ Press. ISBN: 978-0-6911-1763-8.

The book by Heidergott *et al.* includes a tropical model of the Dutch railway network and is more accessible than either the book by Maclagan and Sturmfels or the latter parts of the article by Speyer and Sturmfels.

[1] Maclagan & Sturmfels write: The adjective"tropical" was coined by French mathematicians, notably Jean-Eric Pin, to honor their Brazilian colleague Imre Simon, who pioneered the use of min-plus algebra in optimization theory. There is no deeper meaning to the adjective "tropical". It simply stands for the French view of Brazil.

13.1 All pairs shortest paths

In a previous lecture we used Breadth First Search (BFS) to solve the single-source shortest paths problem in a weighted graph $G(V, E, w)$ where the weights are trivial in the sense that $w(e) = 1 \ \forall e \in E$. Today we'll consider the problem where the weights can vary from edge to edge, but are constrained so that all cycles have positive weight. This ensures that Bellman's equations have a unique solution. Our approach to the problem depends on two main ingredients: a result about powers of the adjacency matrix and a novel kind of arithmetic.

13.2 Counting walks using linear algebra

Our main result is very close in spirit to the following, simpler one.

Theorem 13.1 (Powers of the adjacency matrix count walks)**.** *Suppose $G(V, E)$ is a graph (directed or undirected) on $n = |V|$ vertices and that A is its adjacency matrix. If we define A^ℓ, the ℓ-th matrix power of A, by*

$$A^{\ell+1} = A^\ell A \qquad \text{and} \qquad A^0 = I_n,$$

where $\ell \in \mathbb{N}$, then for $\ell > 0$,

$$A^\ell_{ij} = \text{the number of walks of length } \ell \text{ from vertex } i \text{ to vertex } j, \tag{13.1}$$

where A^ℓ_{ij} is the i, j entry in A^ℓ.

Proof. We'll prove this by induction on ℓ, the number of edges in the walk. The base case is $\ell = 1$ and so $A^\ell = A^1 = A$ and A_{ij} certainly counts the number of one-step walks from vertex i to vertex j: there is either exactly one such walk, or none.

Now suppose the result is true for all $\ell \leq \ell_0$ and consider

$$A^{\ell_0+1}_{ij} = \sum_{k=1}^{n} A^{\ell_0}_{ik} A_{kj}.$$

The only nonzero entries in this sum appear for those values of k for which both $A^{\ell_0}_{ik}$ and A_{kj} are nonzero. Now, the only possible nonzero value for A_{kj} is 1, which happens when the edge (k, j) is present in the graph. Thus we could also think of the sum above as running over vertices k such that the edge (k, j) is in E:

$$A^{\ell_0+1}_{ij} = \sum_{\{k|(k,j)\in E\}} A^{\ell_0}_{ik}.$$

By the inductive hypothesis, $A^{\ell_0}_{ik}$ is the number of distinct, length-ℓ_0 walks from i to k. And if we add the edge (k, j) to the end of such a walk, we get a walk from i to j. All the walks produced in this way are clearly distinct (those that pass through different intermediate vertices k are obviously distinct and even those that have the same k are, by the inductive hypothesis, different somewhere along the i to k segment). Further, every walk of length $\ell_0 + 1$ from i to j must consist of a length-ℓ_0 walk from i to some neighbour k of j, followed by a step from k to j, so we have completed the inductive step. $\square$

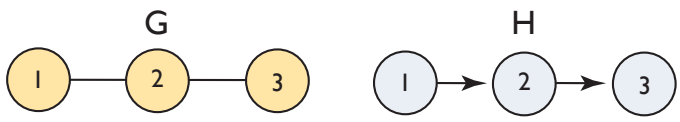

Figure 13.1: *In G, the graph at left, any walk from vertex 1 to vertex 3 must have even length while in H, the directed graph at right, there are no walks of length 3 or greater.*

Two examples

The graph at left in Figure 13.1 contains six walks of length 2. If we represent them with vertex sequences they're

$$(1,2,1),\ (1,2,3),\ (2,1,2),\ (2,3,2),\ (3,2,1),\ \text{ and } (3,2,3), \tag{13.2}$$

while the first two powers of A_G, G's adjacency matrix, are

$$A_G = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad A_G^2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \tag{13.3}$$

Comparing these we see that the computation based on powers of A_G agrees with the list of paths, just as Theorem 13.1 leads us to expect:

- $A^2_{1,1} = 1$ and there is a single walk, (1, 2, 1), from vertex 1 to itself;
- $A^2_{1,3} = 1$: counts the single walk, (1, 2, 3), from vertex 1 to vertex 3;
- $A^2_{2,2} = 2$: counts the two walks from vertex 2 to itself, (2,1,2) and (2,3,2);
- $A^2_{3,1} = 1$: counts the single walk, (3, 2, 1), from vertex 3 to vertex 1;
- $A^2_{3,3} = 1$: counts the single walk, (3, 2, 3), from vertex 3 to itself.

Something similar happens for the directed graph H that appears at right in Figure 13.1, but it has only a single walk of length two and none at all for lengths three or greater.

$$A_H = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad A_H^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A_H^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{13.4}$$

An alternative to BFS

The theorem we've just proved suggests a way to find all the shortest paths in the special case where $w(e) = 1\ \forall e \in E$. Of course, in this case the weight of a path is the same as its length.

(1) Compute the sequence of powers of the adjacency matrix A, A^2, $\cdots$, A^{n-1}.

(2) Observe that a shortest path has length at most $n-1$

(3) To find the length of a shortest path from v_i to v_j, look through the sequence of matrix powers and find the smallest ℓ such that $A^\ell_{ij} > 0$. This ℓ is the desired length.

In the rest of the lecture we'll generalise this strategy to graphs with arbitrary weights.

13.3 Tropical arithmetic

Tropical arithmetic acts on the set $\mathbb{R} \cup \{\infty\}$ and has two binary operations, $\oplus$ and $\otimes$, defined by

$$x \oplus y = \min(x, y) \qquad \text{and} \qquad x \otimes y = x + y \tag{13.5}$$

where, in the definition of $\otimes$, $x + y$ means ordinary addition of real numbers supplemented by the extra rule that $x \otimes \infty = \infty$ for all $x \in \mathbb{R} \cup \{\infty\}$. These novel arithmetic operators have many of the properties familiar from ordinary arithmetic. In particular, they are *commutative*. For all $a, b \in \mathbb{R} \cup \{\infty\}$ we have both

$$a \oplus b = \min(a, b) = \min(b, a) = b \oplus a$$

and

$$a \otimes b = a + b = b + a = b \otimes a.$$

The tropical arithmetic operators are also *associative*:

$$a \oplus (b \oplus c) = \min(a, \min(b, c)) = \min(a, b, c) = \min(\min(a, b), c) = (a \oplus b) \oplus c$$

and

$$a \otimes (b \otimes c) = a + b + c = (a \otimes b) \otimes c.$$

Also, there are distinct additive and multiplicative *identity elements*:

$$a \oplus \infty = \min(a, \infty) = a \qquad \text{and} \qquad b \otimes 0 = 0 + b = b.$$

Finally, the multiplication is *distributive*:

$$a \otimes (b \oplus c) = a + \min(b, c) = \min(a + b,\ a + c) = (a \otimes b) \oplus (a \otimes c).$$

There are, however, important differences between tropical and ordinary arithmetic. In particular, there are no additive inverses[2] in tropical arithmetic and so one cannot always solve linear equations. For example, there is no $x \in \mathbb{R} \cup \{\infty\}$ such that $(2 \otimes x) \oplus 5 = 11$. To see why, rewrite the equation as follows:

$$(2 \otimes x) \oplus 5 = (2 + x) \oplus 5 = \min(2 + x, 5) \leq 5.$$

[2]Students who did Algebraic Structures II might recognise that this collection of properties means that tropical arithmetic over $\mathbb{R} \cup \{\infty\}$ is a *semiring*.

13.3.1 Tropical matrix operations

Given two $m \times n$ matrices A and B whose entries are drawn from $\mathbb{R} \cup \{\infty\}$, we'll define the *tropical matrix sum* $A \oplus B$ by:

$$(A \oplus B)_{ij} = A_{ij} \oplus B_{ij} = \min(A_{ij},\, B_{ij})$$

And for compatibly-shaped tropical matrices A and B we can also define the *tropical matrix product* by

$$(A \otimes B)_{ij} = \bigoplus_{k=1}^{n} A_{ik} \otimes B_{kj} = \min_{1 \le k \le n} (A_{ik} + B_{kj}).$$

Finally, if B is an $n \times n$ square matrix, we can define tropical matrix powers as follows:

$$B^{\otimes k+1} = B^{\otimes k} \otimes B \qquad \text{and} \qquad B^{\otimes 0} = \hat{I}_n. \tag{13.6}$$

where $\hat{I}_n$ is the $n \times n$ *tropical identity matrix*,

$$\hat{I}_n = \begin{bmatrix} 0 & \infty & & \dots & \infty \\ \infty & 0 & \infty & & \vdots \\ & \ddots & \ddots & \ddots & \\ \vdots & & \infty & 0 & \infty \\ \infty & \dots & & \infty & 0 \end{bmatrix}. \tag{13.7}$$

It has zeroes on the diagonal and ∞ everywhere else. It's easy to check that if A is an $m \times n$ tropical matrix then

$$\hat{I}_m \otimes A = A = A \otimes \hat{I}_n.$$

Example 13.2 (Tropical matrix operations)**.** *If we define two tropical matrices A and B by*

$$A = \begin{bmatrix} 1 & 2 \\ 0 & \infty \end{bmatrix} \qquad \textit{and} \qquad B = \begin{bmatrix} \infty & 1 \\ 1 & \infty \end{bmatrix}$$

then

$$A \oplus B \;=\; \begin{bmatrix} 1 \oplus \infty & 2 \oplus 1 \\ 0 \oplus 1 & \infty \oplus \infty \end{bmatrix} = \begin{bmatrix} \min(1, \infty) & \min(2, 1) \\ \min(0, 1) & \min(\infty, \infty) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & \infty \end{bmatrix}$$

and

$$\begin{aligned} A \otimes B &= \begin{bmatrix} (1 \otimes \infty) \oplus (2 \otimes 1) & (1 \otimes 1) \oplus (2 \otimes \infty) \\ (0 \otimes \infty) \oplus (\infty \otimes 1) & (0 \otimes 1) \oplus (\infty \otimes \infty) \end{bmatrix} \\ &= \begin{bmatrix} \min(1 + \infty,\, 2 + 1) & \min(1 + 1,\, 2 + \infty) \\ \min(0 + \infty,\, \infty + 1) & \min(0 + 1,\, \infty + \infty) \end{bmatrix} \\ &= \begin{bmatrix} 3 & 2 \\ \infty & 1 \end{bmatrix}. \end{aligned}$$

13.3.2 A tropical version of Bellman's equations

Recall Bellman's equations from Section 12.3.2. Given a weighted graph $G(V, E, w)$ in which all cycles have positive weight, we can find $u_j = d(v_1, v_j)$ by solving the system

$$u_1 = 0 \qquad \text{and} \qquad u_j = \min_{k \neq j} u_k + w_{k,j} \qquad \text{for } 2 \leq j \leq n,$$

where $w_{k,j}$ is an entry in a weight matrix w given by

$$w_{k,j} \;=\; \begin{cases} w(v_k, v_j) & \text{if } (v_k, v_j) \in E \\ \infty & \text{otherwise} \end{cases} . \tag{13.8}$$

We can rewrite Bellman's equations using tropical arithmetic

$$u_j \;=\; \min_{k \neq j} u_k + w_{k,j} \;=\; \min_{k \neq j} u_k \otimes w_{k,j} \;=\; \bigoplus_{k \neq j} u_k \otimes w_{k,j}$$

which looks almost like the tropical matrix product $u \otimes w$: we'll exploit this observation in the next section.

13.4 Minimal-weight paths in a tropical style

We'll now return to the problem of finding the weights of all the minimal-weight paths in a weighted graph. The calculations are very similar to those in Section 13.2, but now we'll take tropical powers of a weight matrix W whose entries are given by:

$$W_{k,j} \;=\; \begin{cases} 0 & \text{if } j = k \\ w(v_k, v_j) & \text{if } (v_k, v_j) \in E \\ \infty & \text{otherwise} \end{cases} . \tag{13.9}$$

Note that W is very similar to the matrix w defined by Eqn. (13.8): the two differ only along the diagonal, where $w_{ii} = \infty$ for all i, while $W_{ii} = 0$.

Lemma 13.3. *Suppose $G(V, E, w)$ is a weighted graph (directed or undirected) on n vertices. If all the cycles in G have positive weight and a matrix W is defined as in Eqn. (13.9), then the entries in $W^{\otimes \ell}$, the ℓ-th tropical power of W, are such that*

$$W_{ii}^{\otimes \ell} = 0 \textit{ for all } i$$

and, for $i \neq j$, either

$$W_{ij}^{\otimes \ell} = \textit{weight of a minimal-weight walk from } v_i \textit{ to } v_j \textit{ containing at most } \ell \textit{ edges}$$

when G contains such a walk or $W_{ij}^{\otimes \ell} = \infty$ if no such walks exist.

Proof. We proceed by induction on ℓ. The base case is $\ell = 1$ and it's clear that the only length-one walks are the edges themselves, while $W_{ii} = 0$ by construction.

Now suppose the result is true for all $\ell \leq \ell_0$ and consider the case $\ell = \ell_0 + 1$. We will first prove the result for the off-diagonal entries, those for which $i \neq j$. For these entries we have

$$W_{i,j}^{\otimes \ell_0+1} \;=\; \bigoplus_{k=1}^{n} W_{i,k}^{\otimes \ell_0} \otimes W_{k,j} \;=\; \min_{1\leq k\leq n} W_{ik}^{\otimes \ell_0} + W_{k,j} \tag{13.10}$$

and inductive hypothesis says that $W_{i,k}^{\otimes \ell_0}$ is either the weight of a minimal-weight walk from v_i to v_k containing ℓ_0 or fewer edges or, if no such walks exist, $W_{ik}^{\otimes \ell_0} = \infty$. $W_{k,j}$ is given by Eqn. (13.9) and so there are three possibilities for the terms

$$W_{i,k}^{\otimes \ell_0} + W_{k,j} \tag{13.11}$$

that appear in the tropical sum (13.10):

- They are infinite for all values of k, and so direct calculation gives $W_{i,j}^{\otimes \ell_0} = \infty$. This happens when, for each k, we have one or both of the following:
  - $W_{i,k}^{\otimes \ell_0} = \infty$, in which case the inductive hypothesis says that there are no walks of length ℓ_0 or less from v_i to v_k or
  - $W_{k,j} = \infty$ in which case there is no edge from v_k to v_j.

  And since this is true for all k, it implies that there are no walks of length $\ell_0 + 1$ or less that run from v_i to v_j. Thus the lemma holds when $i \neq j$ and $W_{ij}^{\otimes \ell_0+1} = \infty$.

- The expression in (13.11) is finite for at least one value of k, but not for $k = j$. Then we know $W_{ij}^{\otimes \ell_0} = \infty$ and so there are no walks of length ℓ_0 or less running from v_i to v_j. Further,

$$\begin{aligned} W_{i,j}^{\otimes \ell_0+1} &= \min_{k\neq j} W_{i,k}^{\otimes \ell_0} + W_{k,j} \\ &= \min_{\{k \,|\, (v_k,v_j)\in E\}} W_{ij}^{\otimes \ell_0} + w(v_k, v_j). \end{aligned} \tag{13.12}$$

  and reasoning such as we used when discussing Bellman's equations—a minimal-weight walk from v_i to v_j consists of a minimal weight walk from v_i to some neighbour (or, in a digraph, some predecessor) v_k of v_j, plus the edge (v_k, v_j)—means that the (13.12) gives the weight of a minimal-weight walk of length $\ell_0 + 1$ and so the lemma holds here too.

- The expression in (13.11) is finite for case $k = j$ and perhaps also for some $k \neq j$. When $k = j$ we have

$$W_{ik}^{\otimes \ell_0} + W_{k,j} \;=\; W_{ij}^{\otimes \ell_0} + W_{j,j} \;=\; W_{ij}^{\otimes \ell_0} + 0 \;=\; W_{ij}^{\otimes \ell_0},$$

  which, by the inductive hypothesis, is the weight of a minimal-weight walk of length ℓ_0 or less. If there are other values of k for which (13.11) is finite, then they give rise to a sum over neighbours (or, if G is a digraph, over predecessors) such as (13.12), which computes the weight of the minimal-weight walk of length $\ell_0 + 1$. The minimum of this quantity and $W_{ij}^{\otimes \ell_0}$ is then the minimal weight for any walk involving $\ell_0 + 1$ or fewer edges and so the lemma holds in this case too.

Finally, note that reasoning above works for $W_{ii}^{\otimes \ell}$ too: $W_{ii}^{\otimes \ell}$ is the weight of a minimal-weight walk from v_i to itself. And given that any walk from v_i to itself must contain a cycle and that all cycles have positive weight, we can conclude that the tropical sum

$$W_{i,i}^{\otimes \ell_0+1} \;=\; \min_k W_{i,k}^{\otimes \ell_0} + W_{k,i}$$

is minimised by $k = i$, when $W_{i,k}^{\otimes \ell_0} = W_{i,i}^{\otimes \ell_0} = 0$ (by the inductive hypothesis) and $W_{i,i} = 0$ (by construction) so

$$\min_k W_{i,k}^{\otimes \ell_0} + W_{k,i} \;=\; W_{i,i}^{\otimes \ell_0} + W_{i,i} \;=\; 0 + 0 \;=\; 0$$

and the theorem is proven for the diagonal entries too. □

Finally, we can state our main result:

Theorem 13.4 (Tropical matrix powers and shortest paths)**.** *If $G(V, E, w)$ is a weighted graph (directed or undirected) on n vertices in which all the cycles have positive weight, then $d(v_i, v_j)$, the weight of a minimal-weight path from v_i to v_j, is given by*

$$d(v_i, v_j) = W_{i,j}^{\otimes (n-1)} \tag{13.13}$$

Proof. First note that for $i \neq j$, any minimal-weight *walk* from v_i to v_j must actually be a minimal weight *path*. One can prove this by contradiction by noting that any walk that isn't a path must revisit at least one vertex. Say that $v_\star$ is one of these revisited vertices. Then the segment of the walk that runs from the first appearance of $v_\star$ to the second must have positive weight (it's a cycle and all cycles in G have positive weight) and so we can reduce the total weight of the walk by removing this cycle. But this contradicts our initial assumption that the walk had minimal weight.

Combining this with the previous lemma and the observation that a path in G contains at most $n - 1$ edges establishes the result. □

An example

The graph illustrated in Figure 13.2 is small enough that we can just read off the weights of its minimal-weight paths. If we assemble these results into a matrix D whose entries are given by

$$D_{ij} \;=\; \begin{cases} 0 & \text{if } i = j \\ d(v_i, v_j) & \text{if } i \neq j \text{ and } v_j \text{ is reachable from } v_i \\ \infty & \text{otherwise} \end{cases}$$

we get

$$D \;=\; \begin{bmatrix} 0 & -2 & -1 \\ 3 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}.$$

To verify Theorem 13.4 we need only write down the weight matrix and its tropical square, which are

$$W = \begin{bmatrix} 0 & -2 & 1 \\ \infty & 0 & 1 \\ 2 & \infty & 0 \end{bmatrix} \qquad \text{and} \qquad W^{\otimes 2} = \begin{bmatrix} 0 & -2 & -1 \\ 3 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}.$$

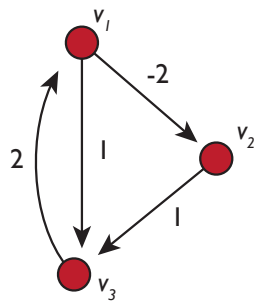

Figure 13.2: *The graph above contains two directed cycles,* (v_1, v_2, v_3, v_1)*, which has weight 1, and* (v_1, v_3, v_1)*, which has weight 2. Theorem 13.4 thus applies and we can compute the weights of minimal-weight paths using tropical powers of the weight matrix.*

The graph has $n = 3$ vertices and so we expect $W^{\otimes(n-1)} = W^{\otimes 2}$ to agree with the distance matrix D, which it does.