Lecture 15

Planar Graphs

This lecture introduces the idea of a planar graph—one that you can draw in such a way that the edges don't cross. Such graphs are of practical importance in, for example, the design and manufacture of integrated circuits as well as the automated drawing of maps. They're also of mathematical interest in that, in a sense we'll explore, there are really only two non-planar graphs.

Reading:
The first part of our discussion is based on that found in Chapter 10 of

> J. A. Bondy and U. S. R. Murty (2008), *Graph Theory*, Vol. 244 of Springer *Graduate Texts in Mathematics*, Springer Verlag,

but in subsequent sections I'll also draw on material from Section 1.5 of

> Dieter Jungnickel (2013), *Graphs, Networks and Algorithms*, 4th edition, which is available online via `SpringerLink`.

15.1 Drawing graphs in the plane

A graph G is said to be *planar* if it is possible to draw it in such a way that the edges intersect only at their end points (the vertices). Such a drawing is also called a *planar diagram* for G or a *planar embedding* of G. Indeed, it is possible to think of such a drawing—call it $\tilde{G}$—as a graph isomorphic to G. Recall our original definition of a graph: it involved only a vertex set V and a set E of pairs of vertices. Take the vertex set of $\tilde{G}$ to be the set of end points of the arcs in the drawing and say that the edge set consists of pairs made up of the two of end points of each arc.

15.1.1 The topology of curves in the plane

To give a clear treatment of this topic, it's helpful to use some ideas from plane topology. That takes us outside the scope of this module and so, in this subsection, I'll give some definitions and state one main result without proof. If you find this material interesting (and it *is* pretty interesting, as well as beautiful and useful) you might consider doing MATH31052, Topology.

Definition 15.1. *A **curve** in the plane is a continuous image of the unit interval. That is, a curve is a set of points*

$$C = \left\{ \gamma(t) \in \mathbb{R}^2 \mid 0 \leq t \leq 1 \right\}$$

*traced out as t varies across the closed unit interval. Here $\gamma(t) = ((x(t),\, y(t)),$ where $x(t) : [0,1] \to \mathbb{R}$ and $y(t) : [0,1] \to \mathbb{R}$ are continuous functions. If the curve does not intersect itself (that is, if $\gamma(t_1) = \gamma(t_2) \Rightarrow t_1 = t_2$) then it is a **simple curve**.*

Definition 15.2. *A **closed curve** is a continuous image of the unit circle or, equivalently, a curve in which $\gamma(0) = \gamma(1)$. If a closed curve doesn't intersect itself anywhere other than at $\gamma(0) = \gamma(1)$, then it is a **simple closed curve**.*

Figure 15.1 and Table 15.1 give examples of these two definitions, while the following one, which is illustrated in Figure 15.2, sets the stage for this section's key result.

Figure 15.1: *From left to right: γ_1, a simple curve; γ_2, a curve that has an intersection, so is not simple; γ_3, a simple closed curve and γ_4, a closed curve with an intersection. Explicit formulae for the curves and their intersections appear in Table 15.1.*

Definition 15.3. *A set $S \subset \mathbb{R}^2$ is **arcwise-connected** if, for every pair of points $x,\, y \in S$, there is a curve $\gamma(t) : [0,1] \to S$ with $\gamma(0) = x$ and $\gamma(1) = y$.*

Theorem 15.4 (The Jordan Curve Theorem)**.** *A simple closed curve C in the plane divides the rest of the plane into two disjoint, arcwise-connected, open sets. These two open sets are called the **interior** and **exterior** of C, often denoted $\mathrm{Int}(C)$ and $\mathrm{Ext}(C)$, and any curve joining a point $x \in \mathrm{Int}(C)$ to a point $y \in \mathrm{Ext}(C)$ intersects C at least once.*

This is illustrated in Figure 15.3.

Curve	$x(t)$	$y(t)$
$\gamma_1(t)$	$2t$	$24t^3 - 36t^2 + 14t - 1$
$\gamma_2(t)$	$24t^3 - 36t^2 + 14t - 1$	$8t^2 - 8t + 1$
$\gamma_3(t)$	$\cos(2\pi t)$	$\sin(2\pi t)$
$\gamma_4(t)$	$\sin(4\pi t)$	$\sin(2\pi t)$

Table 15.1: *Explicit formulae for the curves appearing in Figure 15.1. The intersection in* γ_2 *occurs at* $\gamma_2\left(\frac{1}{2} - \sqrt{\frac{1}{6}}\right) = \gamma_2\left(\frac{1}{2} + \sqrt{\frac{1}{6}}\right) = \left(0, \frac{1}{3}\right)$, *while the one for* γ_4 *happens where* $\gamma_4(0) = \gamma_4\left(\frac{1}{2}\right) = (0, 0)$.

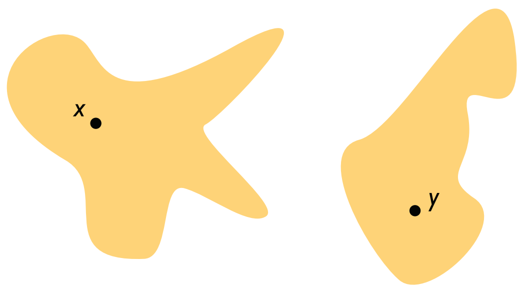


Figure 15.2: *The two shaded regions below are, individually, arcwise connected, but their union is not: any curve connecting* x *to* y *would have to pass outside the shaded regions.*

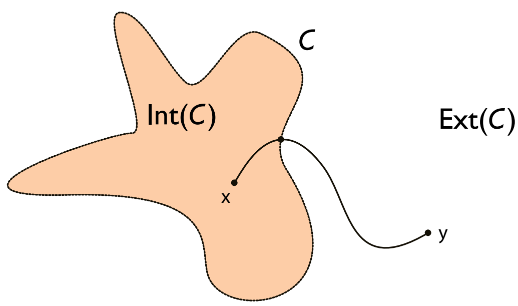


Figure 15.3: *An illustration of the Jordan Curve Theorem.*

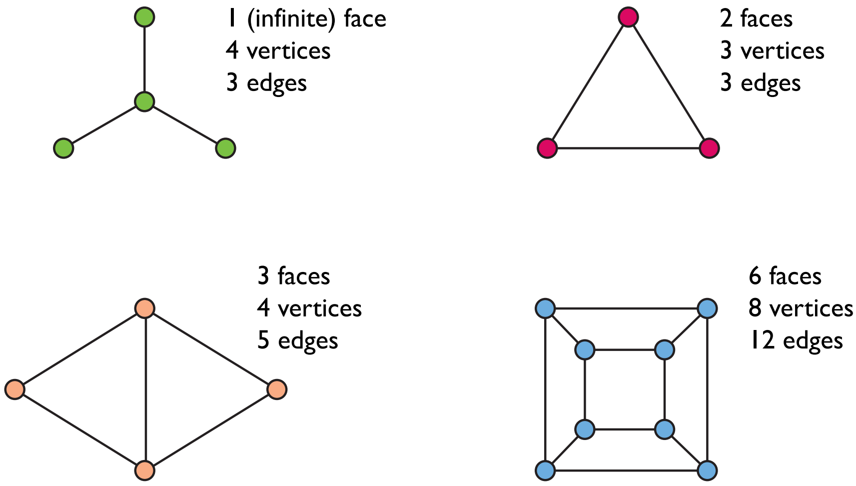


Figure 15.4: *Four examples of planar graphs, with numbers of faces, vertices and edges for each.*

15.1.2 Faces of a planar graph

The definitions in the previous section allow us to be a bit more formal about the definition of a planar graph:

Definition 15.5. *A **planar diagram** for a graph $G(V, E)$ with edge set $E = \{e_1, \ldots, e_m\}$ is a collection of simple curves $\{\gamma_1, \ldots, \gamma_m\}$ that represent the edges and have the property that the curves γ_j and γ_k corresponding to two distinct edges e_j and e_k intersect if and only if the two edges are incident on the same vertex and, in this case, they intersect only at the endpoints that correspond to their common vertex.*

Definition 15.6. *A graph G is **planar** if and only if it has a planar diagram.*

If a planar graph G contains cycles then the curves that correspond to the edges in the cycles link together to form simple closed curves that divide the plane into finitely many disjoint open sets called *faces*. Even if the graph has no cycles, there will still be one *infinite face*: see Figure 15.4.

15.2 Euler's formula for planar graphs

Our first substantive result about planar graphs is:

Theorem 15.7 (Euler's formula)**.** *If $G(V, E)$ is a connected planar graph with $n = |V|$ vertices and $m = |E|$ edges, then any planar diagram for G has $f = 2+m-n$ faces.*

Before giving a full proof, we begin with an easy special case:

Lemma 15.8 (Euler's formula for trees)**.** *If $G(V, E)$ is a tree then $f = 2 + m - n$.*

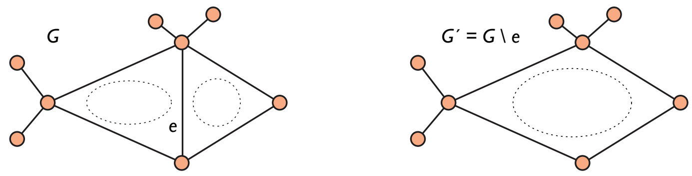


Figure 15.5: *Deleting the edge e causes two adjacent faces in G to merge into a single face in G'.*

Proof of the lemma about trees: As G is a tree we know $m = n - 1$, so

$$2 + m - n \;=\; 2 + (n-1) - n \;=\; 2 - 1 \;=\; 1 \;=\; f,$$

where the last equality follows because every planar diagram for a tree has only a single, infinite face. $\square$

Proof of Euler's formula in the general case: We'll prove the result for arbitrary connected planar graphs by induction on m, the number of edges.

Base case The smallest connected planar graph contains only a single vertex, so has $n = 1$, $m = 0$ and $f = 1$. Thus

$$2 + m - n = 2 + 0 - 1 = 1 = f$$

just as Euler's formula demands.

Inductive step Suppose the result is true for all $m \leq m_0$ and consider a connected planar graph $G(V, E)$ with $|E| = m = m_0 + 1$ edges. Also suppose that G has n vertices and a planar diagram with f faces. Then one of the following things is true:

- G is a tree, in which case Euler's formula is true by the lemma proved above;
- G contains at least one cycle.

If G contains a cycle, choose an edge $e \in E$ that's part of that cycle and form $G' = G \backslash e$, which has $m' = m_0$ edges, $n' = n$ vertices and $f' = f - 1$ faces. This last follows because breaking a cycle merges two adjacent faces, as is illustrated in Figure 15.5.

As G' has only m_0 edges, we can use the inductive hypothesis to say that $f' = m' - n' + 2$. Then, again using unprimed symbols for quantities in G, we have:

$$\begin{aligned} f' &= m' - n' + 2 \\ f - 1 &= m_0 - n + 2 \\ f &= (m_0 + 1) - n + 2 \\ f &= m - n + 2, \end{aligned}$$

which establishes Euler's formula for graphs that contain cycles.

$\square$

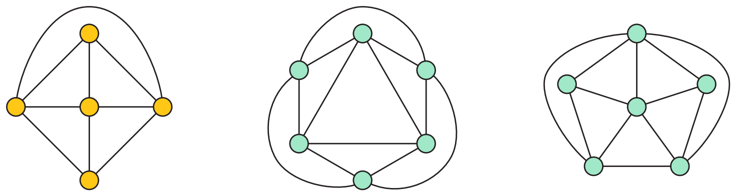


Figure 15.6: *Planar graphs with the maximal number of edges for a given number of vertices. The graph with the yellow vertices has $n = 5$ and $m = 9$ edges, while those with the blue vertices have $n = 6$ and $m = 12$*

15.3 Planar graphs can't have many edges

To set the scene for our next result, consider graphs on $n \in \{1, 2, \ldots, 5\}$ vertices and, for each n, try to draw a planar graph with as many edges as possible. At first this is easy: it's possible to find a planar diagram for each of the complete graphs K_1, K_2, K_3 and K_4, but, as we will prove below, K_5 is not planar and the the best one can do is to find a planar graph with $n = 5$ and $m = 9$. For $n = 6$ there are two non-isomorphic planar graphs with $m = 12$ edges, but none with $m \geq 12$. Figure 15.6 shows examples of planar graphs having the maximal number of edges.

Larger planar graphs (those with $n \gg 5$) tend to be even *sparser*, which means that they have many fewer edges than they could. The relevant comparison for a graph on n vertices is $n(n-1)/2$, the number of edges in the complete graph K_n, so we'll say that a graph is *sparse* if

$$|E| \ll \frac{n(n-1)}{2} \qquad \text{or} \qquad s \equiv \frac{|E|}{n(n-1)/2} \ll 1 \tag{15.1}$$

Table 15.2 makes it clear that when $n > 5$ the planar graphs become increasingly sparse[1].

15.3.1 Preliminaries: bridges and girth

The next two definitions will help us to formulate and prove our main result, a somewhat technical theorem that gives a precise sense to the intuition that a planar graph can't have very many edges.

Definition 15.9. *An edge e in a connected graph $G(V, E)$ is a **bridge** if the graph $G' = G \backslash e$ formed by deleting e has more than one connected component.*

Definition 15.10. *If a graph $G(V, E)$ contains one or more cycles then the **girth** of G is the length of a shortest cycle.*

These definitions are illustrated in Figures 15.7 and 15.8.

[1] I wrote software to compute the first few rows of this table myself, but got the counts for $n > 9$ from the *On-Line Encyclopedia of Integer Sequences*, entries `A003094` and `A001349`.

n	m_{max}	s	Number of non-isomorphic, connected graphs that are … planar, with $m = m_{max}$	planar	planar or not
5	9	0.9	1	20	21
6	12	0.8	2	99	112
7	15	0.714	5	646	853
8	18	0.643	14	5,974	11,117
9	21	0.583	50	71,885	261,080
10	24	0.583	?	1,052,805	11,716,571
11	27	0.533	?	17,449,299	1,006,700,565
12	30	0.491	?	313,372,298	164,059,830,476

Table 15.2: *Here m_{max} is the maximal number of edges appearing in a planar graph on the given number of vertices, while the column labelled s lists the measure of sparsity given by Eqn. 15.1 for connected, planar graphs with m_{max} edges. The remaining columns list counts of various kinds of graphs and make the point that as n increases, planar graphs with $m = m_{max}$ become rare in the set of all connected planar graphs and that this family itself becomes rare in the family of connected graphs.*

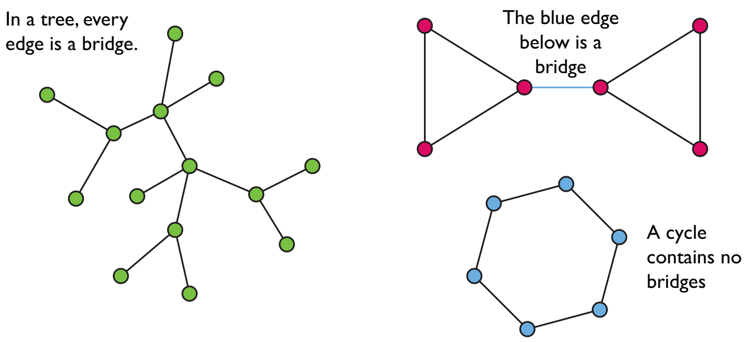


Figure 15.7: *Several examples of bridges in graphs.*

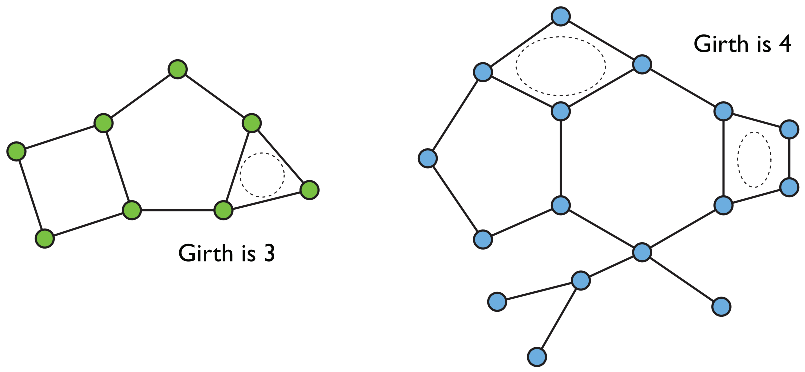


Figure 15.8: *The girth of a graph is the length of a shortest cycle.*

Remark 15.11. *A graph with n vertices has girth in the range $3 \leq g \leq n$. The lower bound arises because all cycles include at least three edges and the upper one because the longest possible cycle occurs when G is isomorphic to C_n.*

15.3.2 Main result: an inequality relating n and m

We are now in a position to state our main result:

Theorem 15.12 (Jungnickel's 1.5.3)**.** *If $G(V, E)$ is a connected planar graph with $n = |V|$ vertices and $m = |E|$ edges then either:*

A: G is acyclic and $m = n - 1$;

B: G has at least one cycle and so has a well-defined girth g. In this case

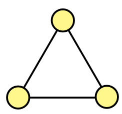
$$m \leq \frac{g(n-2)}{g-2}. \tag{15.2}$$

Outline of the Proof. We deal first with the case where G is acyclic and then move on to the harder, more general case:

A: G is connected, so if it has no cycles it's a tree and we've already proved (see Theorem 6.13) that trees have $m = n - 1$.

B: When G contains one or more cycles, we'll prove the inequality 15.2 mainly by induction on n, but we'll need several sub-cases. To see why, let's plan out the argument.

Base case: $n = 3$
There is only a single graph on three vertices that contains a cycle, it's K_3,



which has girth $g = 3$ and $n = 3$, so our theorem says

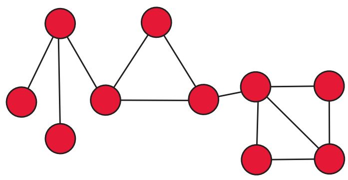
$$\begin{aligned} m &\leq \frac{g(n-2)}{g-2} \\ &\leq \frac{3 \times (3-2)}{3-2} \\ &\leq 3 \end{aligned}$$

which is obviously true.

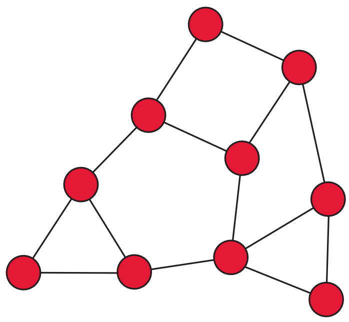
Inductive hypothesis:
Assume the result is true for all connected, planar graphs that contain a cycle and have $n \leq n_0$ vertices.

Inductive step:
Now consider a connected, planar graph $G(V, E)$ with $n_0 + 1$ vertices that contains a cycle. We need, somehow, to reduce this graph to one for which we can exploit the inductive hypothesis and so one naturally thinks of deleting something. This leads to two main sub-cases, which are illustrated[2] below.

B.1 G contains at least one bridge. In this case the road to a proof by induction seems clear: we'll delete the bridge and break G into two smaller graphs.



B.2 G does not contains any bridges. Equivalently, every edge in G is part of some cycle. Here it's less clear how to handle the inductive step and so we will use an altogether different, non-inductive approach.



We'll deal with these cases in turn, beginning with **B.1**.

As mentioned above, a natural approach is to delete a bridge and break G into two smaller graphs—say, $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$—then apply the inductive

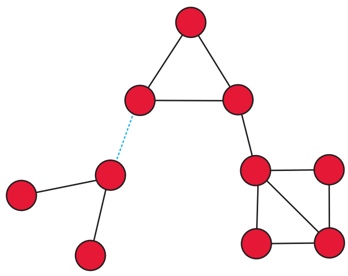
[2]The examples illustrating cases **B.1** and **B.2** are meant to help the reader follow the argument, but are *not* part of the logic of the proof.

hypothesis to the pieces. If we define $n_j = |V_j|$ to be the number of vertices in G_j and $m_j = |E_j|$ to be the corresponding number of edges, then we know

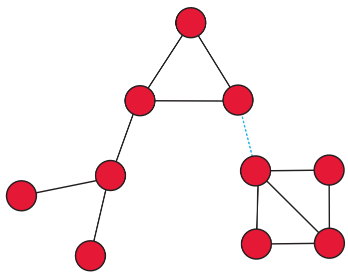
$$n_1 + n_2 = n \qquad \text{and} \qquad m_1 + m_2 = m - 1. \tag{15.3}$$

But we need to take a little care as deleting a bridge leads to two further sub-cases and we'll need a separate argument for each. Given that the original graph G contained at least one cycle—and noting that removing a bridge can't break a cycle—we know that at least one of the two pieces G_1 and G_2 contains a cycle. Our two sub-cases are thus:

B.1a Exactly one of the two pieces contains a cycle. We can assume without loss of generality that it's G_2, so that G_1 is a tree.



B.1b Both G_1 and G_2 contain cycles.



Thus we can complete the proof of Theorem 15.12 by producing arguments (full details below) that cover the following three possibilities

B.1a G contains a bridge and at least one cycle. Deleting the bridge leaves two subgraphs, a tree G_1 and a graph, G_2, that contains a cycle: we handle this possibility in Case 15.13 below.

B.1b G contains a bridge and at least two cycles. Deleting the bridge leaves two subgraphs, G_1 and G_2, each of which contains at least one cycle: see Case 15.14.

B.2 G contains one or more cycles, but no bridges: see Case 15.15.

□

15.3.3 Gritty details of the proof of Theorem 15.12

Before we plunge into the Lemmas, it's useful to make a few observations about the ratio $g/(g-2)$ that appears in Eqn. (15.2). Recall (from Remark 15.11) that if a graph on n vertices contains a cycle, then the girth is well-defined and lies in the range $3 \leq g \leq n$.

- For $g > 2$, the ratio $g/(g-2)$ is a monotonically decreasing function of g and so
$$g_1 > g_2 \Rightarrow \left(\frac{g_1}{g_1-2}\right) < \left(\frac{g_2}{g_2-2}\right). \tag{15.4}$$
- The monotonicity of $g/(g-2)$, combined with the fact that $g \geq 3$, implies that $g/(g-2)$ is bounded from above by 3:
$$g \geq 3 \Rightarrow \left(\frac{g}{g-2}\right) \leq \left(\frac{3}{3-2}\right) = 3. \tag{15.5}$$
- And at the other extreme, $g/(g-2)$ is bounded from below (strictly) by 1:
$$g \leq n \Rightarrow \left(\frac{g}{g-2}\right) \geq \left(\frac{n}{n-2}\right) > 1. \tag{15.6}$$

The three cases

The cases below are all part of an inductive argument in which, $G(V,E)$ is a connected planar graph with $|V| = n_0 + 1$ and $|E| = m$. It also contains at least one cycle and so has a well-defined girth, g. Finally, we have an inductive hypothesis saying that Theorem 15.12 holds for all trees and for all connected planar graphs with $|V| \leq n_0$.

Case 15.13 (Case **B.1a** of Theorem 15.12)**.** *Here G contains a bridge and deleting this bridge breaks G into two connected planar, subgraphs, $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$, one of which is a tree.*

Proof. We can assume without loss that G_1 is the tree and then argue that every cycle that appears in G is also in G_2 (we've only deleted a bridge), so the girth of G_2 is still g. Also, $n_1 \geq 1$, so $n_2 \leq n_0$ and, by the inductive hypothesis, we have

$$m_2 \;\leq\; \frac{g(n_2-2)}{g-2}.$$

But then, because G_1 is a tree, we know that $m_1 = n_1 - 1$. Adding this to both sides of the inequality yields

$$m_1 + m_2 \;\leq\; (n_1 - 1) + \frac{g(n_2-2)}{g-2}$$

or, equivalently,

$$m_1 + m_2 + 1 \ \le \ n_1 + \frac{g(n_2 - 2)}{g - 2}.$$

Finally, noting that $m = m_1 + m_2 + 1$, we can say

$$\begin{aligned} m &\le n_1 + \frac{g(n_2 - 2)}{g - 2} \\ &\le \left(\frac{g}{g-2}\right) n_1 + \frac{g(n_2 - 2)}{g - 2} \\ &\le \frac{g(n_1 + n_2 - 2)}{g - 2} \\ &\le \frac{g(n - 2)}{g - 2}, \end{aligned}$$

which is the result we sought. Here the step from the first line to the second follows because $1 < g/(g - 2)$ (recall Eqn. (15.6)), so

$$n_1 < \left(\frac{g}{g-2}\right) n_1$$

and the last line follows because $n = n_1 + n_2$. □

Case 15.14 (Case **B.1b** of Theorem 15.12)**.** *This case is similar to the previous one in that here again G contains a bridge, but in this case deleting the bridge breaks G into two connected planar, subgraphs, each of which contains at least one cycle (and so has a well defined-girth).*

Proof. We'll say that G_1 has girth g_1 and G_2 has girth g_2 and note that, as the girth is defined as the length of a *shortest* cycle—and as every cycle that appears in the original graph G must still be present in one of the G_j—we know that

$$g \le g_1 \qquad \text{and} \qquad g \le g_2. \tag{15.7}$$

Now, $n = n_0 + 1$ and $n = n_1 + n_2$ so as we know that $n_j \ge 3$ (the shortest possible cycle is of length 3 and the G_j contain cycles), it follows that we have both $n_1 < n_0$ and $n_2 < n_0$. This means that the inductive hypothesis applied to both G_j and so we have

$$m_1 \le \frac{g_1(n_1 - 2)}{g_1 - 2} \qquad \text{and} \qquad m_2 \le \frac{g_2(n_2 - 2)}{g_2 - 2}.$$

Adding these together yields:

$$\begin{aligned} m_1 + m_2 &\le \frac{g_1(n_1 - 2)}{g_1 - 2} + \frac{g_2(n_2 - 2)}{g_2 - 2} \\ &\le \frac{g(n_1 - 2)}{g - 2} + \frac{g(n_2 - 2)}{g - 2} \\ &\le \frac{g(n_1 + n_2 - 4)}{g - 2}, \end{aligned}$$

where the step from the first line to the second follows from Eqn. 15.7 and the monotonicity of the ratio $g/(g-2)$ (recall Eqn. (15.4)). If we again note that $1 < g/(g-2)$ we can conclude that

$$\begin{aligned} m_1 + m_2 + 1 &\leq \frac{g(n_1+n_2-4)}{g-2} + 1 \\ &\leq \frac{g(n_1+n_2-4)}{g-2} + \frac{g}{g-2} \\ &\leq \frac{g(n_1+n_2-3)}{g-2} \end{aligned}$$

and so

$$m = m_1 + m_2 + 1 \;\leq\; \frac{g(n_1+n_2-3)}{g-2} \;\leq\; \frac{g(n_1+n_2-2)}{g-2}$$

or, as $n = n_1 + n_2$,

$$m \leq \frac{g(n-2)}{g-2},$$

which is the result we sought. □

Case 15.15 (Case **B.2** of Theorem 15.12)**.** *In the final case $G(V, E)$ does not contain any bridges, which implies that every edge in E is part of some cycle. This makes it harder to see how to use the inductive hypothesis (we'd have to delete two or more edges to break G into disconnected pieces ...) and so we will use an entirely different argument based on Euler's Formula (Theorem 15.7).*

Proof. First, define f_j to be the number of faces whose boundary has j edges, making sure to include the infinite face: Figures 15.9 illustrates this definition. Then, as each edge appears in the boundary of exactly two faces we have both

$$\sum_{j=g}^{n} f_j = f \qquad \text{and} \qquad \sum_{j=g}^{n} j \times f_j = 2m.$$

Note that both sums start at g, the girth, as we know that there are no cycles of shorter length. But then

$$2m \;=\; \sum_{j=g}^{n} j \times f_j \;\geq\; \sum_{j=g}^{n} g \times f_j \;=\; g \sum_{j=g}^{n} f_j \;=\; gf,$$

where we obtain the inequality by replacing the length of the cycle j in $j \times f_j$ with g, the length of the shortest cycle (and hence the smallest value of j for which f_j is nonzero). Thus we have

$$2m \geq gf \qquad \text{or} \qquad f \leq 2m/g.$$

If we now use Euler's Theorem to say that $f = m - n + 2$, we have

$$m - n + 2 \;\leq\; \frac{2m}{g} \qquad \text{or} \qquad m - \frac{2m}{g} \;\leq\; n - 2.$$

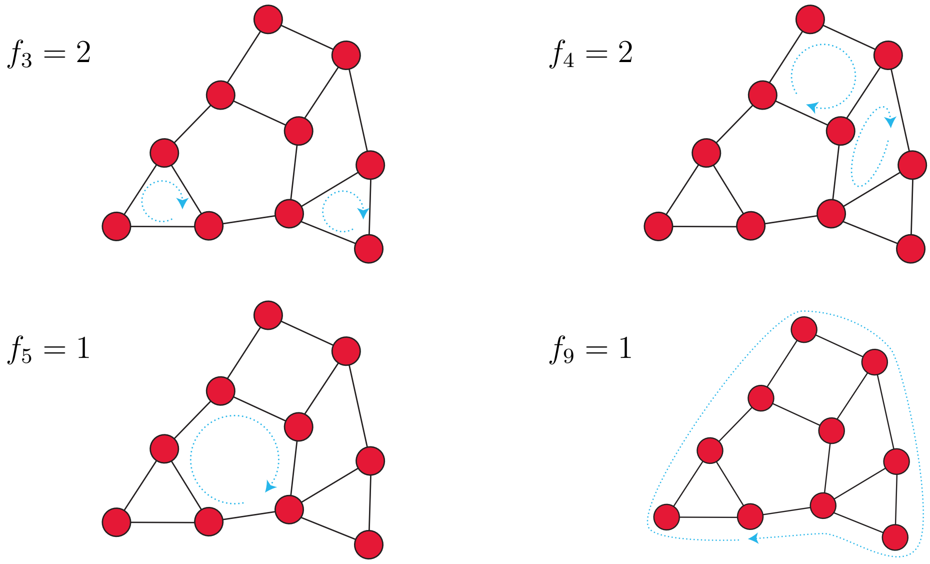


Figure 15.9: *The example used to illustrate case **B.2** of Theorem 15.12 has $f_3 = 2$, $f_4 = 2$, $f_5 = 1$ and $f_9 = 1$ (for the infinite face): all other f_j are zero.*

And then, finally,

$$\frac{gm}{g} - \frac{2m}{g} \leq n - 2 \quad \text{so} \quad \frac{(g-2)m}{g} \leq n - 2 \quad \text{and} \quad m \leq \frac{g(n-2)}{g-2}$$

which is the result we sought.

□

15.3.4 The maximal number of edges in a planar graph

Theorem 15.12 has an easy corollary that gives a simple bound on the maximal number of edges in a graph with $|V| = n$.

Corollary 15.16. *If $G(V, E)$ is a connected planar graph with $n = |V| \geq 3$ vertices and $m = |E|$ edges then $m \leq 3n - 6$.*

Proof. Either G is a tree, in which case $m = n - 1$ and the bound in the Corollary is certainly satisfied, or G contains at least one cycle. In the latter case, say that the girth of G is g. We know $3 \leq g \leq n$ and our main result says

$$m \leq \left(\frac{g}{g-2} \right) (n-2).$$

Thus, recalling that $g/(g-2) \leq 3$, the result follows immediately. □

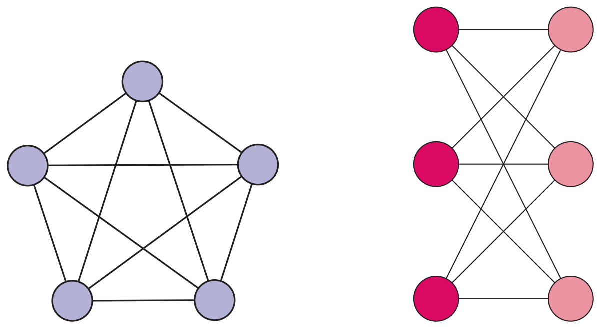


Figure 15.10: *Both K_5 and $K_{3,3}$ are non-planar.*

15.4 Two non-planar graphs

The hard-won inequalities from the previous section—which both say something like "G planar implies m small"—cannot be used to prove that a graph *is* planar[3], but can help establish that a graph *isn't*. The idea is to use the contrapositives, which are statements like "If m is too big, then G can't be planar."

To illustrate this, we'll use our inequalities to prove that neither of the graphs in Figure 15.10—K_5 at left and $K_{3,3}$ at right—is planar. Let's begin with K_5: it has $n = 5$ so Corollary 15.16 says that if it is planar,

$$m \leq 3n - 6 = 3 \times 5 - 6 = 15 - 6 = 9,$$

but K_5 actually has $m = 10$ edges, which is one too many for a planar graph. Thus K_5 can't have a planar diagram.

$K_{3,3}$ isn't planar either, but Corollary 15.16 isn't strong enough to establish this. $K_{3,3}$ has $n = 6$ and $m = 3 \times 3 = 9$. Thus it easily satisfies the bound from Corollary 15.16, which requires only that $m \leq 3 \times 6 - 6 = 12$. But if we now apply our main result, Theorem 15.12, we'll see that $K_{3,3}$ can't be planar. The relevant inequality is

$$\begin{aligned} m &\leq \frac{g(n-2)}{g-2} \\ &\leq \frac{4 \times (6-2)}{4-2} \\ &\leq \frac{16}{2} \\ &\leq 8 \end{aligned}$$

where, in passing from the first line to the second, I've used the fact that the girth of $K_{3,3}$ is $g = 4$. To see this, first note that any cycle in a bipartite graph has even

[3] There is an $O(n)$ algorithm that determines whether a graph on n vertices is planar and, if it is, produces a planar diagram. We don't have time to discuss it, but interested readers might like to look at John Hopcroft and Robert Tarjan (1974), Efficient Planarity Testing, *Journal of the ACM*, **21**(4):549–568. DOI: 10.1145/321850.321852

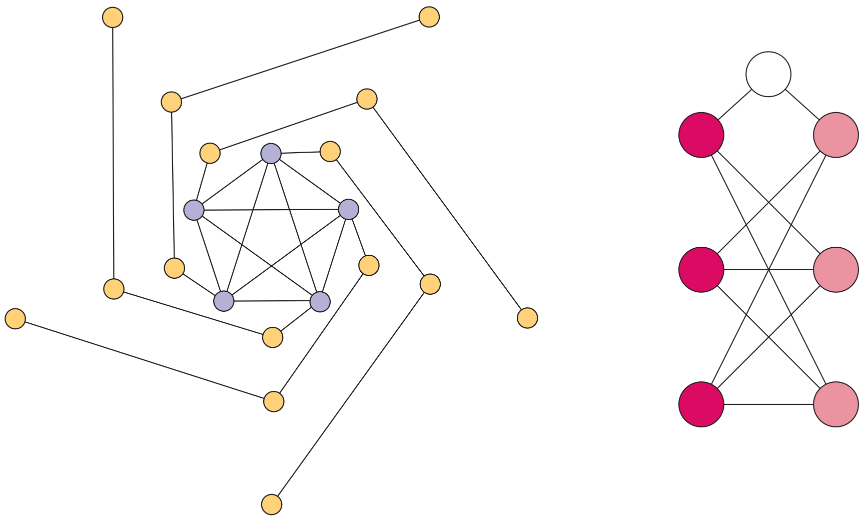


Figure 15.11: *Knowing that K_5 and $K_{3,3}$ are non-planar makes it clear that these two graphs can't be planar either, even though neither violates the inequalities from the previous section (check this).*

length, so the shortest possible cycle in $K_{3,3}$ has length 4, and then find such a cycle (there are lots).

Once we know that $K_{3,3}$ and K_5 are nonplanar, we can see immediately that many other graphs must be non-planar too, even when this would not be detected by either of our inequalities: Figure 15.11 shows two such examples. The one on the left has K_5 as a subgraph, so even though it satisfies the bound from Theorem 15.12, it can't be planar. The example at right is similar in that any planar diagram for this graph would obviously produce a planar diagram for $K_{3,3}$, but the sense in which this second graph "contains" $K_{3,3}$ is more subtle: we'll clarify and formalise this in the next section, then state a theorem that says, essentially, that *every* non-planar graph contains K_5 or $K_{3,3}$.

15.5 Kuratowski's Theorem

We begin with a pair of definitions designed to capture the sense in which the graph at right in Figure 15.11 contains $K_{3,3}$.

Definition 15.17. *A* ***subdivision*** *of a graph $G(V, E)$ is a graph $H(V', E')$ formed by (perhaps repeatedly) removing an edge $e = (a, b) \in E$ from G and replacing it with a path*

$$\{(a, v_1), (v_1, v_2), \ldots, (v_k, b)\}$$

containing of some number $k > 0$ of new vertices $\{v_1, \ldots, v_k\}$, each of which has degree 2.

Figure 15.12 shows a couple examples of subdivisions, including one at left that gives an indication of where the name comes from: the extra vertices can be thought of as dividing an existing edge into smaller ones.

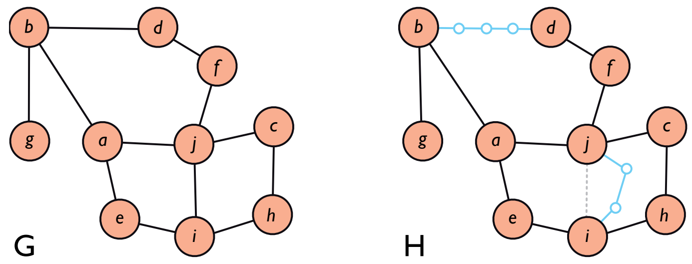


Figure 15.12: *H at right is a subdivision of G. The connection between b and d, which was a single edge in G, becomes a blue path in H: one can imagine that the original edge (b, d) has had three new, white vertices inserted into it, "sub-dividing" it. The other deleted edge, (i, j) is shown as a pale grey, dashed line (to indicate that it's not part of H), while the new path that replaces it is again shown in blue and white.*

Definition 15.18. *Two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are said to be* ***homeomorphic*** *if they are isomorphic to subdivisions of the same graph.*

That is, we say G_1 and G_2 are homeomorphic if there is some third graph—call it G_0—such that both G_1 and G_2 are subdivisions of G_0. Figure 15.13 shows several graphs that are homeomorphic to K_5. Homeomorphism is an equivalence relation on graphs[4] and so all the graphs in Figure 15.13 are homeomorphic to each other as well as to K_5.

The notion of homeomorphism allows us to state the following remarkable result:

Theorem 15.19 (Kuratowski's Theorem (1930))**.** *A graph G is planar if and only if it does not contain a subgraph homeomorphic to K_5 or $K_{3,3}$.*

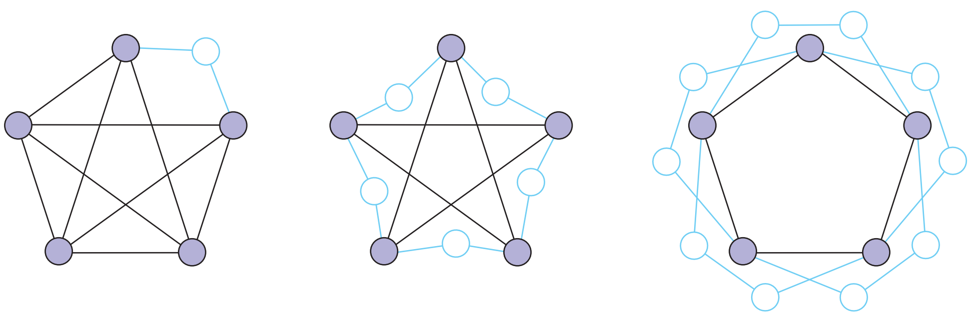


Figure 15.13: *These three graphs are homeomorphic to K_5, and hence also to each other.*

[4]The keen reader should check this for herself.

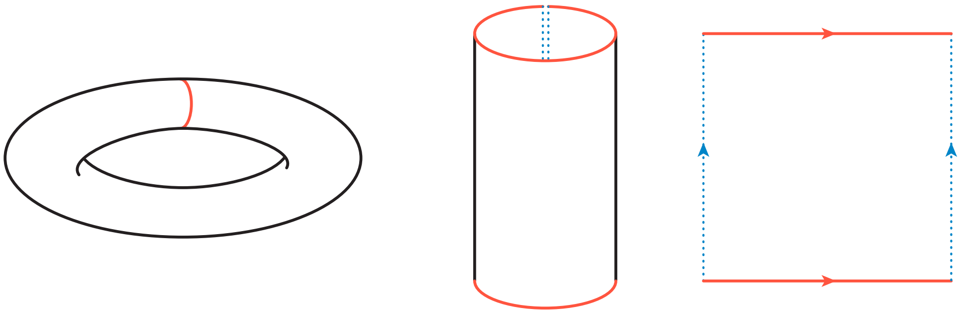


Figure 15.14: *The two-torus cut twice and flattened into a square.*

15.6 Afterword

The fact that there are, in a natural sense, only two non-planar graphs is one of the main reasons we study the topic. But this turns out to be the easiest case of an even more amazing family of results that I'll discuss briefly. These other theorems have to do with drawing graphs on arbitrary surfaces (spheres, tori ...)—it's common to refer to this as *embedding* the graph in the surface—and the process uses curves similar to those discussed in Section 15.1.1, except that now we want, for example, curves $\gamma : [0, 1] \to S_2$, where S_2 is the two-sphere, the surface of a three-dimensional unit ball.

Embedding a graph in the sphere turns out to be the same as embedding it in the plane: you can imagine drawing the planar diagram on a large, thin, stretchy sheet and then smoothing it onto a big ball in such a way that the diagram lies in the northern hemisphere while the edges of the sheet are all drawn together in a bunch at the south pole. Similarly, if we had a graph embedded in the sphere we could get a planar diagram for it by punching a hole in the sphere. Thus a graph can be embedded in the sphere unless it contains—in the sense of Kuratowski's Theorem—a copy of K_5 or $K_{3,3}$. For this reason, these two graphs are called *topological obstructions* to embedding a graph in the plane or sphere. They are also sometimes referred to as *forbidden subgraphs*.

But if we now consider the torus, the situation for K_5 and $K_{3,3}$ is different. To make drawings, I'll use a standard representation of the torus as a square: you should imagine this square to have been peeled off a more familiar torus-as-a-doughnut, as illustrated in Figure 15.14. Figure 15.15 then shows embeddings of K_5 ad $K_{3,3}$ in the torus—these are analogous to planar diagrams in that the arcs representing the edges don't intersect except at their endpoints.

There are, however, graphs that one cannot embed in the torus and there is even an analog of Kuratowski's Theorem that says that there are finitely many forbidden subgraphs and that all non-toroidal[5] graphs include at least one of them. In fact, something even more spectacular is true: early in an epic series[6] of papers,

[5]By analogy with the term non-planar, a graph is said to be *non-toroidal* if it cannot be embedded in the torus.

[6]The titles all begin with the words "Graph Minors". The series began in 1983 with "Graph Minors. I. Excluding a forest" (DOI: 10.1016/0095-8956(83)90079-5) and seems

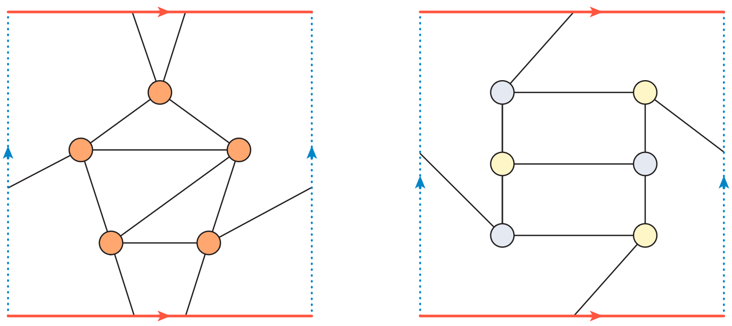


Figure 15.15: *Embeddings of K_5 (left) and $K_{3,3}$ (right) in the torus. Edges that run off the top edge of the square return on the bottom, while those that run off the right edge come back on the left.*

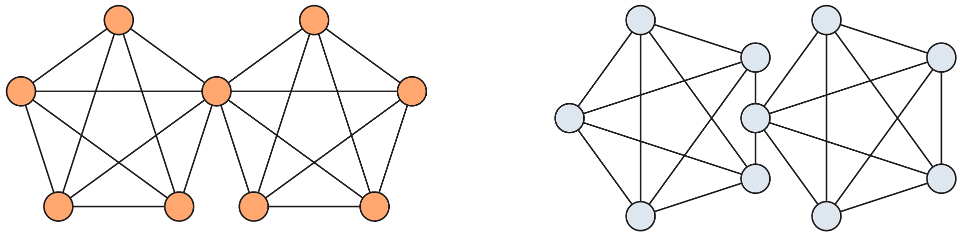


Figure 15.16: *Neither of these graphs can be embedded in the two-torus. These examples come from Andrei Gargarin, Wendy Myrvold and John Chambers (2009), The obstructions for toroidal graphs with no $K_{3,3}$'s,* Discrete Mathematics, ***309****(11):3625–3631. DOI: 10.1016/j.disc.2007.12.075*

Neil Robertson and Paul D. Seymour proved that *every* surface (the sphere, the torus, the torus with two holes...) has a Kuratowski-like theorem with a finite list of forbidden subgraphs: two of those for the torus are shown in Figure 15.16. One shouldn't, however, draw too much comfort from the word "finite". In her recent MSc thesis[7] Ms. Jennifer Woodcock developed a new algorithm for embedding graphs in the torus and tested it against a database that, although known to be incomplete, includes 239,451 forbidden subgraphs.

to have concluded with "Graph Minors. XXIII. Nash-Williams' immersion conjecture" in 2010 (DOI: 10.1016/j.jctb.2009.07.003). The result about embedding graphs in surfaces appeared in 1990 in "Graph Minors. VIII. A Kuratowski theorem for general surfaces" (DOI: 10.1016/0095-8956(90)90121-F).

[7]Ms. Woodcock's thesis, *A Faster Algorithm for Torus Embedding*, is lovely and is the source of much of the material in this section.