Lecture 9

Proof of Tutte's Matrix-Tree Theorem

The proof here is derived from a terse account in the lecture notes from a course on Algebraic Combinatorics taught by Lionel Levine at MIT in Spring 2011.[1] I studied them with Samantha Barlow, a former Discrete Maths student who did a third-year project with me in 2011-12.

Reading:
I don't know of any textbook accounts of the proof given here, but the intrepid reader might like to look at the following two articles, both of which make the connection between the Principle of Inclusion/Exclusion and Tutte's Matrix Tree theorem.

- J.B. Orlin (1978), Line-digraphs, arborescences, and theorems of Tutte and Knuth, *Journal of Combinatorial Theory, Series B*, **25**(2):187–198. DOI: 10.1016/0095-8956(78)90038-2
- S. Chaiken (1983), A combinatorial proof of the all minors matrix tree theorem, *SIAM Journal on Algebraic and Discrete Methods*, **3**:319–329. DOI: 10.1137/0603033

9.1 Single predecessor graphs

Before we plunge into the proof itself I'd like to define a certain family of graphs that includes, but is larger than, the family of spanning arborescences.

Definition 9.1. *A* ***single predecessor graph ("spreg") with distinguished vertex*** *v in a digraph $G(V, E)$ is a subgraph $T(V, E')$ (Note that T and G have the same vertex set) in which each vertex other than the distinguished vertex v has exactly one predecessor while v itself has no predecessors. Equivalently,*

$$\deg_{in}(v) = 0 \qquad \text{and} \qquad \deg_{in}(u) = 1 \; \forall u \neq v \in V.$$

[1]Dr. Levine seems to have moved to a post at Cornell, but his notes were still available via the link above in January 2020.

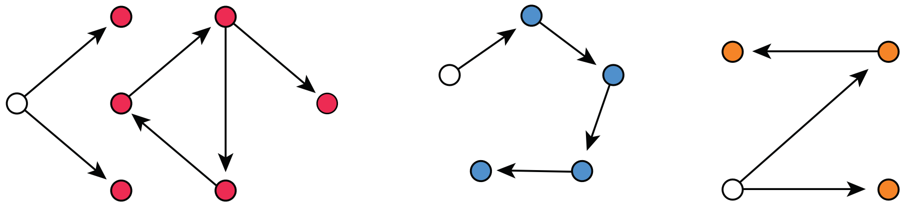

Figure 9.1: *Three examples of* ***s****ingle* ***pre****decessor* ***g****raphs (spregs). In each the distinguished vertex is white, while the other vertices, which all have* $\deg_{in}(u) = 1$*, are shaded in other colours. The example at left, has multiple weakly connected components, while the other two are arborescences.*

Figure 9.1 includes several examples of spregs, including two that are arborescences, which prompts the following proposition:

Proposition 9.2 (Spanning arborescences are spregs)**.** *If* $T(V, E')$ *is a spanning arborescence for* $G(V, E)$ *with root* v*, then it is also a spreg with distinguished vertex* v*.*

Proof. By definition, G and T share the same vertex set, so all we need check is that the vertices $u \neq v$ in T have a single predecessor. Recall that an arborescence rooted at v is a directed graph $T(V, E)$ such that

(i) Every vertex $u \neq v$ is accessible form v. That is, there is a directed path from v to every other vertex.

(ii) T becomes an ordinary, undirected tree if we ignore the directedness of the edges.

The proposition consists of two separate claims: that $\deg_{in}(v) = 0$ and that $\deg_{in}(u) = 1 \ \forall u \neq v \in V$. We'll prove both by contradiction.

Suppose that $\deg_{in}(v) > 0$: it's then easy to see that T must include a directed cycle. Consider one of v's predecessors—call is u_0. It is accessible from v, so there is a directed path from v to u_0. And u_0 is a predecessor of v, so there is also a directed edge $(u_0, v) \in E$. If we append this edge to the end of the path, we get a directed path from v back to itself. This contradicts the second property of an arborescence and so we must have $\deg_{in}(v) = 0$.

The proof for the second part of the proposition is illustrated in Figure 9.2. Suppose that $\exists \ u \neq v \in V$ such that $\deg_{in}(u) \geq 2$ and choose two distinct predecessors of u: call them v_1 and v_2 and note that one of them may be the root vertex v. Now consider the directed paths from v to v_1 and v_2. In the undirected version of T these paths, along with the edges (v_1, u) and (v_2, u), must include a cycle, which contradicts the second property of an arborescence. □

The examples in Figure 9.1 make it clear that there are other kinds of spregs besides spanning arborescences, but there aren't *that* many kinds:

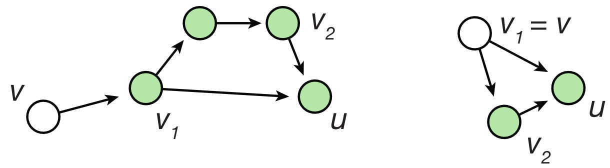

Figure 9.2: *Two examples to illustrate the second part of the proof that an arborescence is a spreg. If one ignores the directedness of the edges in the graphs above, both contain cycles.*

Proposition 9.3 (Characterising spregs)**.** *A spreg with distinguished vertex v consists of an arborescence rooted at v, plus zero or more disjoint weakly connected components, each of which contains a single directed cycle.*

Note that the arborescence mentioned in the Proposition is not necessarily a spanning one: the leftmost graph in Figure 9.1 consists of a small, non-spanning arborescence and a second component that contains a cycle.

The reasoning needed to prove this proposition is similar to that for the previous one and so is left to the Problem Sets.

This lemma is one of the key ingredients in the proof of Tutte's Matrix-Tree Theorem. The idea is to first note that a spanning arborescence is a spreg. We then count the spanning arborescences contained in a graph by first counting *all* the spregs, then use the Principle of Inclusion/Exclusion to count—and subtract away—those spregs that contain one or more cycles.

9.2 Counting spregs with determinants

Recall that we're trying to prove

Theorem 1 (Tutte's Directed Matrix-Tree Theorem, 1948)**.** *If $G(V, E)$ is a digraph with vertex set $V = \{v_1, \ldots, v_n\}$ and L is an $n \times n$ matrix whose entries are given by*

$$L_{ij} = \begin{cases} \deg_{in}(v_j) & \text{If } i = j \\ -1 & \text{If } i \neq j \text{ and } (v_i, v_j) \in E \\ 0 & \text{Otherwise} \end{cases} \tag{9.1}$$

then the number N_j of spanning arborescences with root at v_j is

$$N_j = \det(\hat{L}_j)$$

where $\hat{L}_j$ is the matrix produced by deleting the j-th row and column from L.

First note that—because we can always renumber the vertices before we apply the theorem—it is sufficient to prove the result for the case with root vertex $v = v_n$. Now consider the representation of $\det(\hat{L}_n)$ as a sum over permutations:

$$\det(\hat{L}_n) \equiv \det(\mathcal{L}) = \sum_{\sigma \in S_{n-1}} \operatorname{sgn}(\sigma) \prod_{j=1}^{n-1} \mathcal{L}_{j\sigma(j)}. \tag{9.2}$$

Predecessor of v_1	v_2	v_3	Is a spanning arboresence?
v_2	v_1	v_2	No
v_2	v_3	v_2	No
v_2	v_4	v_2	Yes
v_4	v_1	v_2	Yes
v_4	v_3	v_2	No
v_4	v_4	v_2	Yes

Table 9.1: *Each row here corresponds to one of the spregs in Figure 9.3.*

where I have introduced the notation $\mathcal{L} \equiv \hat{L}_n$ to avoid the confusion of having two kinds of subscripts on $\hat{L}_n$. This means that $\mathcal{L}$ is an $(n-1) \times (n-1)$ matrix in which

$$\mathcal{L}_{ij} = L_{ij},$$

where L_{ij} is the i, j entry in the matrix L defined by Eqn. (9.1) in the statement of Tutte's theorem.

9.2.1 Counting spregs

In this section we'll explore two examples that illustrate a connection between terms in the sum for $\det(\mathcal{L})$ and the business of counting various kinds of spregs.

The identity term: counting all spregs

In the case where $\sigma = \text{id}$, so that $\sigma(j) = j$ for all j, we have $\text{sgn}(\sigma) = 1$ and

$$\prod_{j=1}^{n-1} \mathcal{L}_{j\sigma(j)} = \prod_{j=1}^{n-1} \mathcal{L}_{jj} = \prod_{j=1}^{n-1} \deg_{in}(v_j). \tag{9.3}$$

This product is also equal to the total number of spregs in $G(V, E)$ that have distinguished vertex v_n. To see why, look back at the definition of a spreg and think about what we'd need to do if we wanted to write down a complete list of these spregs. We could specify a spreg by listing the single predecessor for each vertex other than v_n in a table like the one below

Vertex	v_1	v_2	v_3
Predecessor	v_2	v_1	v_2

which describes one of the spregs rooted at v_4 contained in the four-vertex graph shown in Figure 9.3. And if we wanted to list *all* the four-vertex spregs contained in this graph we could start by assembling the predecessor lists of all the vertices other than the distinguished vertex,

$$P_1 = \{v_2, v_4\}, \; P_2 = \{v_1, v_3, v_4\} \text{ and } P_3 = \{v_2\},$$

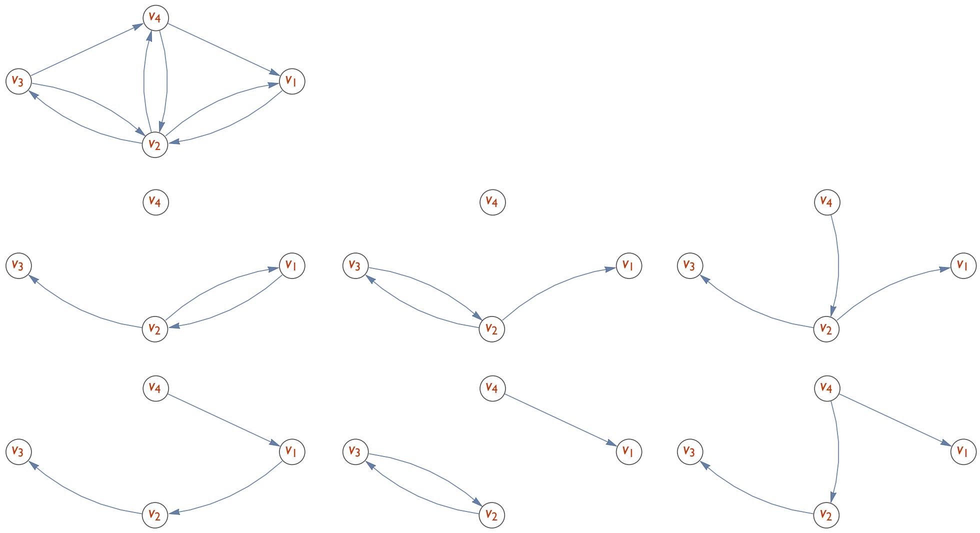

Figure 9.3: *The graph $G(V, E)$ at upper left contains six spregs with distinguished vertex v_4, all of which are shown in the two rows below. Three of them are spanning arborescences rooted at v_4, while the three others contain cycles.*

where P_j lists the predecessors of v_j. Then, to specify a spreg with distinguished vertex v_4 we would choose one entry from each of the predecessor lists, meaning that there are

$$|P_1| \times |P_2| \times |P_3| = \deg_{in}(v_1) \times \deg_{in}(v_2) \times \deg_{in}(v_3) = 2 \times 3 \times 1 = 6$$

such spregs in total. All six possibilities are listed in Table 9.1 and illustrated in Figure 9.3. The equation above also emphasises that $|P_j| = \deg_{in}(v_j)$ and so makes the connection with the product in Eqn. (9.3).

Terms that count spregs containing a single directed cycle

In the case where the permutation σ contains a single cycle of length ℓ, so that

$$\sigma = (i_1, \ldots, i_\ell),$$

we have $\text{sgn}(\sigma) = (-1)^{\ell-1}$ and

$$\begin{aligned} \prod_{j=1}^{n-1} \mathcal{L}_{j\sigma(j)} &= \left(\prod_{j \in \text{fix}(\sigma)} \mathcal{L}_{jj} \right) \times \left(\prod_{k=1}^{\ell} \mathcal{L}_{i_k i_{k+1}} \right) \\ &= \left(\prod_{j \in \text{fix}(\sigma)} \deg_{in}(v_j) \right) \times \left(\prod_{k=1}^{\ell} \mathcal{L}_{i_k i_{k+1}} \right) \end{aligned}$$

where the indices i_k are to be understood periodically, so $i_{\ell+1} = i_1$. The factors $\mathcal{L}_{i_k i_{k+1}}$ in the second of the two products above are off-diagonal entries of $\mathcal{L} = \hat{L}_n$ and thus satisfy

$$\mathcal{L}_{i_k i_{k+1}} = \begin{cases} -1 & \text{if } (v_{i_k}, v_{i_{k+1}}) \in E \\ 0 & \text{otherwise.} \end{cases}$$

Thus if one or more of the edges $(v_{i_k}, v_{i_{k+1}})$ is absent from the graph we have

$$\prod_{k=1}^{\ell} \mathcal{L}_{i_k i_{k+1}} = 0,$$

but if all the edges $(v_{i_k}, v_{i_{k+1}})$ are present we have can make the following observations:

- the graph contains a directed cycle given by the vertex sequence

$$(v_{i_1},\, \ldots,\, v_{i_\ell},\, v_{i_1});$$

- $\mathcal{L}_{i_k i_{k+1}} = -1$ for all $1 \leq k \leq \ell$ and so we have

$$\begin{aligned} \operatorname{sgn}(\sigma) \prod_{j=1}^{n-1} \mathcal{L}_{j\sigma(j)} &= (-1)^{\ell-1} \left(\prod_{j \in \operatorname{fix}(\sigma)} \deg_{in}(v_j) \right) \times \prod_{k=1}^{\ell} (-1) \\ &= (-1)^{\ell-1} \left(\prod_{j \in \operatorname{fix}(\sigma)} \deg_{in}(v_j) \right) (-1)^{\ell} \\ &= (-1)^{2\ell-1} \prod_{j \in \operatorname{fix}(\sigma)} \deg_{in}(v_j) \\ &= - \prod_{j \in \operatorname{fix}(\sigma)} \deg_{in}(v_j). \end{aligned} \tag{9.4}$$

Arguments similar to those in the previous section then show that the product $\prod_{j \in \operatorname{fix}(\sigma)} \deg_{in}(v_j)$ in Eqn. (9.4) counts the number of ways to choose predecessors for those vertices that aren't part of the cycle. We can summarise all these ideas with the following pair of results:

Proposition 9.4. *For a permutation $\sigma \in S_{n-1}$ consisting of a single cycle*

$$\sigma = (i_1,\, \ldots,\, i_\ell)$$

define an associated directed cycle C_σ specified by the vertex sequence $(v_{i_1},\, \ldots,\, v_{i_\ell},\, v_{i_1})$. Then the term in $\det(\mathcal{L})$ corresponding to σ satisfies

$$\operatorname{sgn}(\sigma) \prod_{j=1}^{n-1} \mathcal{L}_{j\sigma(j)} \;=\; \begin{cases} -\displaystyle\prod_{j \in \operatorname{fix}(\sigma)} \deg_{in}(v_j) & \textit{if } C_\sigma \subseteq G(V,E) \textit{ and } \operatorname{fix}(\sigma) \neq \emptyset \\ -1 & \textit{if } C_\sigma \subseteq G(V,E) \textit{ and } \operatorname{fix}(\sigma) = \emptyset \\ 0 & \textit{if } C_\sigma \not\subseteq G(V,E) \end{cases}$$

Corollary 9.5. *For σ and C_σ as in Proposition 9.4*

$$\left| \prod_{j=1}^{n-1} \mathcal{L}_{j\sigma(j)} \right| \;=\; \left| \{ \textit{spregs containing } C_\sigma \} \right|.$$

σ	C_σ	$C_\sigma \subseteq G$?	$\left\lvert\prod_{j=1}^{n-1} \mathcal{L}_{j\sigma(j)}\right\rvert$	Number of spregs containing C_σ
(1,2)	(v_1, v_2, v_1)	Yes	$\deg_{in}(v_3) = 1$	1
(1,3)	(v_1, v_3, v_1)	No	$\deg_{in}(v_2) \times 0$	0
(2,3)	(v_2, v_3, v_2)	Yes	$\deg_{in}(v_1) = 2$	2
(1,2,3)	(v_1, v_2, v_3, v_1)	No	0	0
(1,3,2)	(v_1, v_3, v_2, v_1)	No	0	0

Table 9.2: *The results of using Corollary 9.5 to count spregs containing the various cycles C_σ associated with the non-identity elements of S_3. The right column lists the number one gets by direct counting of the spregs shown in Figure 9.3.*

9.2.2 An example

Before pressing on to generalise the results of the previous section to arbitrary permutations, let's see what Corollary 9.5 allows us to say about the graph in Figure 9.3. There $G(V, E)$ is a digraph on four vertices, so the determinant that comes into Tutte's theorem is that of L_4, a three-by-three matrix. We've already seen that if $\sigma = \text{id}$ the product $\prod_{j \in \text{fix}(\sigma)} \deg_{in}(v_j)$ gives six, the total number of spregs contained in the graph. The results for the remaining elements of S_3 are listed in Table 9.2 and all are covered by Corollary 9.5, as all non-identity elements of S_3 are single cycles.

9.2.3 Counting spregs in general

Here we generalise the results from Section 9.2.1 to permutations that are the products of arbitrarily many cycles.

Lemma 9.6 (Counting spregs containing cycles)**.**
Suppose $\sigma \in S_{n-1}$ is the product of $k > 0$ disjoint cycles

$$\sigma = (i_{1,1}, \dots, i_{1,\ell_1}) \dots (i_{k,1}, \dots, i_{k,\ell_k}),$$

where ℓ_j is the length of the j-th cycle. Associate the directed cycle C_j defined by the vertex sequence $(v_{i_{j,1}}, \dots, v_{i_{j,\ell_j}}, v_{i_{j,1}})$ with the j-th cycle in the permutation and define

$$C_\sigma = \bigcup_{j=1}^{k} C_j.$$

Then the term in $\det(\mathcal{L})$ corresponding to σ satisfies

$$\text{sgn}(\sigma) \prod_{j=1}^{n-1} \mathcal{L}_{j\sigma(j)} = \begin{cases} (-1)^k \prod_{j \in \text{fix}(\sigma)} \deg_{in}(v_j) & \text{if } C_\sigma \subseteq G(V,E) \text{ and } \text{fix}(\sigma) \neq \emptyset \\ (-1)^k & \text{if } C_\sigma \subseteq G(V,E) \text{ and } \text{fix}(\sigma) = \emptyset \\ 0 & \text{if } C_\sigma \not\subseteq G(V,E) \end{cases}$$

Further,

$$\left|\prod_{j=1}^{n-1} \mathcal{L}_{j\sigma(j)}\right| = \left|\left\{ \textit{spregs containing } C_\sigma = \bigcup_{j=1}^{k} C_j \right\}\right|. \tag{9.5}$$

The proof of this result requires reasoning much like that used in Section 9.2.1 and so is left to the reader.

9.3 Proof of Tutte's theorem

Throughout this section I will continue to write $\mathcal{L}$ in place of $\hat{L}_n$ to avoid a confusing welter of subscripts.

Proof. As we argued at the beginning of Section 9.2, it is sufficient to prove that $\det(\hat{L}_n) = \det(\mathcal{L})$ is the number of spanning arborescences rooted at v_n. We'll do this with the Principle of Inclusion/Exclusion and so, to begin, we need to specify the universal set U and the subsets X_j. Begin by considering the set $\mathcal{C}$ of all possible directed cycles involving the vertices $v_1 \dots v_{n-1}$. It's clearly a finite set and so we can declare that it has M elements and imagine that we've chosen some (arbitrary) numbering scheme so that we can list the set of cycles as

$$\mathcal{C} = \{C_1, \dots, C_M\}.$$

We'll then choose the sets U and X_j as follows:

- U is the set of all spregs with distinguished vertex v_n. That is, U is the set of subgraphs of $G(V, E)$ in which

$$\deg_{in}(v_n) = 0 \qquad \text{and} \qquad \deg_{in}(v_j) = 1 \text{ for } 1 \leq j \leq (n-1).$$

- $X_j \subseteq U$ is the subset of U consisting of spregs containing the cycle C_j. This subset may, of course, be empty.

Proposition 9.3—the one about characterising spregs—tells us that a spreg that has distinguished vertex v_n is either a spanning arboresence rooted at v_n or a graph that contains one or more disjoint cycles. This means that

$$N_n = |\{\text{spanning arborescences rooted at } v_n\}| = |U| - \left|\bigcup_{j=1}^{M} X_j\right|.$$

and the Principle of Inclusion/Exclusion then says

$$\begin{aligned} N_n &= |U| - \left(\sum_{I \subseteq \{1,\dots,M\},\, I \neq \emptyset} (-1)^{|I|-1} \left| \bigcap_{j \in I} X_j \right| \right) \\ &= |U| + \sum_{I \subseteq \{1,\dots,M\},\, I \neq \emptyset} (-1)^{|I|} \left| \bigcap_{j \in I} X_j \right| \end{aligned} \tag{9.6}$$

As we know that spregs contain only disjoint cycles, we can say

$$|X_j \cap X_k| = 0 \text{ unless } C_j \cap C_k = \emptyset$$

and so can eliminate many of the terms in the sum over intersections in Eqn. (9.6), rewriting it as a sum over collections of disjoint cycles:

$$N_n = |U| + \sum_{\substack{I \subseteq \{1,\ldots,M\},\, I \neq \emptyset \\ C_j \cap C_k = \emptyset \;\; \forall j \neq k \in I}} (-1)^{|I|} \left| \bigcap_{j \in I} X_j \right|. \tag{9.7}$$

Then we can use the lemma from the previous section—Lemma 9.6, which relates non-identity permutations to numbers of spregs containing cycles—to rewrite Eqn. (9.7) in terms of permutations. First note that Eqn. (9.5) allows us to write

$$\left| \bigcap_{j \in I} X_j \right| = \left| \left\{ \text{spregs containing } \bigcup_{j \in I} C_j \right\} \right| = \left| \prod_{k=1}^{n-1} \mathcal{L}_{k\sigma_I(k)} \right|.$$

Here $\sigma_I \in S_{n-1}$ is the permutation

$$\sigma_I = \prod_{j \in I} \sigma_{C_j}$$

whose cycle representation is the product of the permutations corresponding to the directed cycles C_j for $j \in I$. In the product above σ_{C_j} is the cycle permutation corresponding to the directed cycle C_j. The correspondence here somes from the bijection between permutations and unions of directed cycles that we discussed in Section 8.2.

Now, again using Lemma 9.6, we have

$$N_n = |U| + \sum_{\substack{I \subseteq \{1,\ldots,M\},\, I \neq \emptyset \\ C_j \cap C_k = \emptyset \;\; \forall j \neq k \in I}} (-1)^{|I|} \left| \prod_{j=1}^{n-1} \mathcal{L}_{j\sigma_I(j)} \right|$$

$$N_n = |U| + \sum_{\substack{I \subseteq \{1,\ldots,M\},\, I \neq \emptyset \\ C_j \cap C_k = \emptyset \;\; \forall j \neq k \in I}} \operatorname{sgn}(\sigma_I) \prod_{j=1}^{n-1} \mathcal{L}_{j\sigma_I(j)} \tag{9.8}$$

As the sum in Eqn. (9.8) ranges over *all* collections of disjoint cycles, the permutations σ_I range over all non-identity permutations in S_{n-1} and so we have

$$N_n = |U| + \sum_{\sigma \neq \text{id}} \operatorname{sgn}(\sigma) \prod_{j=1}^{n-1} \mathcal{L}_{j\sigma(j)}. \tag{9.9}$$

Finally, from Eqn. (9.3) we know that

$$|U| = |\{\text{spregs containing all } v \in V \text{ with distinguished vertex } v_n\}| = \prod_{j=1}^{n-1} \deg_{in}(v_j)$$

which is the term in $\det(\mathcal{L})$ corresponding to the identity permutation. Combining this observation with Eqn. (9.9) gives us

$$N_n \;=\; \sum_{\sigma \in S_{n-1}} \operatorname{sgn}(\sigma) \prod_{j=1}^{n-1} \mathcal{L}_{j\sigma(j)} \;=\; \det(\mathcal{L}),$$

which is the result we sought.

$\square$