

Lecture 7

The Matrix-Tree Theorems

This section of the notes introduces a pair of very beautiful theorems that use linear algebra to count trees in graphs.

Reading:

The next few lectures are not covered in Jungnickel's book, though a few definitions in our Section 7.2.1 come from his Section 1.6. But the main argument draws on ideas that you should have met in Foundations of Pure Mathematics, Linear Algebra and Algebraic Structures.

7.1 Kirchoff's Matrix-Tree Theorem

Our goal over the next few lectures is to establish a lovely connection between Graph Theory and Linear Algebra. It is part of a circle of beautiful results discovered by the great German physicist Gustav Kirchoff in the mid-19th century, when he was studying electrical circuits. To formulate his result we need a few new definitions.

Definition 7.1. A subgraph $T(V, E')$ of a graph $G(V, E)$ is a **spanning tree** if it is a tree that contains every vertex in V .

Figure 7.1 gives some examples.

Definition 7.2. If $G(V, E)$ is a graph on n vertices with $V = \{v_1, \dots, v_n\}$ then its **graph Laplacian** L is an $n \times n$ matrix whose entries are

$$L_{ij} = \begin{cases} \deg(v_j) & \text{If } i = j \\ -1 & \text{If } i \neq j \text{ and } (v_i, v_j) \in E \\ 0 & \text{Otherwise} \end{cases}$$

Equivalently, $L = D - A$, where D is a diagonal matrix with $D_{jj} = \deg(v_j)$ and A is the graph's adjacency matrix.

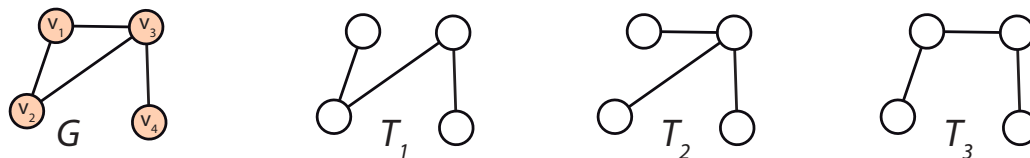


Figure 7.1: A graph $G(V, E)$ with $V = \{v_1, \dots, v_4\}$ and three of its spanning trees: T_1, T_2 and T_3 . Note that although T_1 and T_3 are isomorphic, we regard them as **different** spanning trees for the purposes of the Matrix-Tree Theorem.

Example 7.3 (Graph Laplacian). The graph G whose spanning trees are illustrated in Figure 7.1 has graph Laplacian

$$\begin{aligned}
 L &= D - A \\
 &= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \tag{7.1}
 \end{aligned}$$

Once we have these two definitions it's easy to state the Matrix-Tree theorem

Theorem 7.4 (Kirchoff's Matrix-Tree Theorem, 1847). If $G(V, E)$ is an undirected graph and L is its graph Laplacian, then the number N_T of spanning trees contained in G is given by the following computation.

- (1) Choose a vertex v_j and eliminate the j -th row and column from L to get a new matrix \hat{L}_j ;
- (2) Compute

$$N_T = \det(\hat{L}_j). \tag{7.2}$$

The number N_T in Eqn. (7.2) counts spanning trees that are distinct as **subgraphs of G** : equivalently, we regard the vertices as distinguishable. Thus some of the trees that contribute to N_T may be isomorphic: see Figure 7.1 for an example.

This result is remarkable in many ways—it seems amazing that the answer doesn't depend on which vertex we choose when constructing \hat{L}_j —but to begin with let's simply use the theorem to compute the number of spanning trees for the graph in Example 7.3

Example 7.5 (Counting spanning trees). If we take G to be the graph whose Laplacian is given in Eqn. (7.1) and choose $v_j = v_1$ we get

$$\hat{L}_1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

and so the number of spanning trees is

$$\begin{aligned}
 N_T &= \det(\hat{L}_1) \\
 &= 2 \times \det \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} - (-1) \times \det \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \\
 &= 2 \times (3 - 1) + (-1 - 0) \\
 &= 4 - 1 = 3
 \end{aligned}$$

I'll leave it as an exercise for the reader to check that one gets the same result from $\det(\hat{L}_2)$, $\det(\hat{L}_3)$ and $\det(\hat{L}_4)$.

7.2 Tutte's Matrix-Tree Theorem

We'll prove Kirchoff's theorem as a consequence of a much more recent result¹ about directed graphs. To formulate this we need a few more definitions that generalise the notion of a tree to digraphs.

7.2.1 Arborescences: directed trees

Recall the definition of *accessible* from Lecture 5:

In a directed graph $G(V, E)$ a vertex b is said to be **accessible** from another vertex a if G contains a walk from a to b . Additionally, we'll say that all vertices are accessible from themselves.

This allows us to define the following suggestive term:

Definition 7.6. A vertex $v \in V$ in a directed graph $G(V, E)$ is a **root** if every other vertex is accessible from v .

We'll then be interested in the following directed analogue of a tree:

Definition 7.7. A directed graph $T(V, E)$ is a **directed tree** or **arborescence** if

- (i) T contains a root
- (ii) The graph $|T|$ that one obtains by ignoring the directedness of the edges is a tree.

See Figure 7.2 for an example. Of course, it's then natural to define an analogue of a spanning tree:

Definition 7.8. A subgraph $T(V, E')$ of a digraph $G(V, E)$ is a **spanning arborescence** if T is an arborescence that contains all the vertices of G .

¹Proved by Bill Tutte about a century after Kirchoff's result in W.T. Tutte (1948), The dissection of equilateral triangles into equilateral triangles, *Math. Proc. Cambridge Phil. Soc.*, **44**(4):463–482.

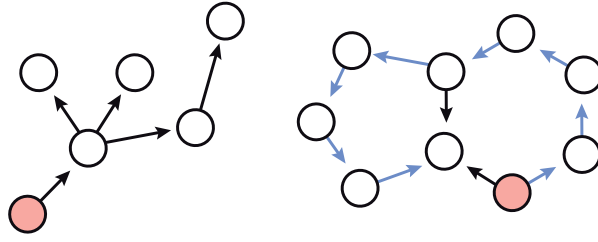


Figure 7.2: The graph at left is an arborescence whose root vertex is shaded red, while the graph at right contains a spanning arborescence whose root is shaded red and whose edges are blue.

7.2.2 Tutte's theorem

Theorem 7.9 (Tutte's Directed Matrix-Tree Theorem, 1948). *If $G(V, E)$ is a digraph with vertex set $V = \{v_1, \dots, v_n\}$ and L is an $n \times n$ matrix whose entries are given by*

$$L_{ij} = \begin{cases} \deg_{in}(v_j) & \text{If } i = j \\ -1 & \text{If } i \neq j \text{ and } (v_i, v_j) \in E \\ 0 & \text{Otherwise} \end{cases} \quad (7.3)$$

then the number N_j of spanning arborescences with root at v_j is

$$N_j = \det(\hat{L}_j) \quad (7.4)$$

where \hat{L}_j is the matrix produced by deleting the j -th row and column from L .

Here again, the number N_j in Eqn. (7.4) counts spanning arborescences that are distinct as **subgraphs of G** : equivalently, we regard the vertices as distinguishable. Thus some of the arborescences that contribute to N_j may be isomorphic, but if they involve different edges we'll count them separately.

Example 7.10 (Counting spanning arborescences). *First we need to build the matrix L defined by Eqn. (7.3) in the statement of Tutte's theorem. If we choose G to be the graph pictured at upper left in Figure 7.3 then this is $L = D_{in} - A$ where D_{in} is a diagonal matrix with $D_{jj} = \deg_{in}(v_j)$ and A is the graph's adjacency matrix.*

$$\begin{aligned} L &= D_{in} - A \\ &= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & -1 \\ -1 & -1 & 0 & 2 \end{bmatrix} \end{aligned}$$

Then Table 7.1 summarises the results for the number of rooted trees.

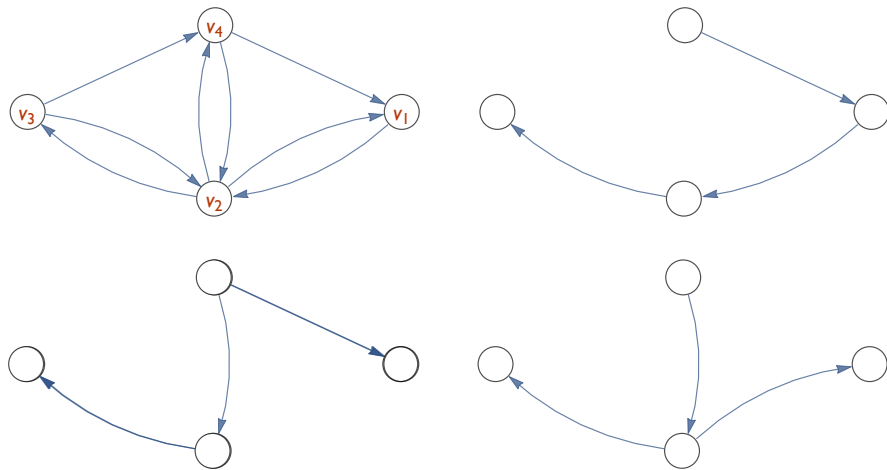


Figure 7.3: The digraph at upper left, on which the vertices are labelled, has three spanning arborescences rooted at v_4 .

j	\hat{L}_j	$\det(\hat{L}_j)$
1	$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix}$	2
2	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix}$	4
3	$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix}$	7
4	$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$	3

Table 7.1: The number of spanning arborescences for the four possible roots in the graph at upper left in Figure 7.3.

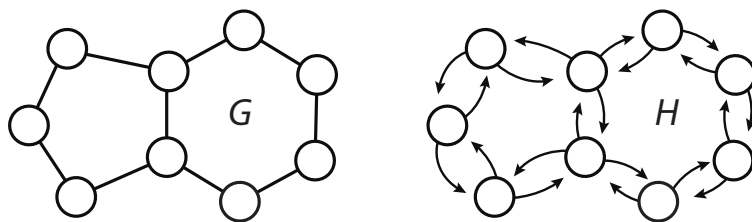


Figure 7.4: If we convert an undirected graph such as G at left to a directed graph such as H at right, it is easy to count the spanning trees in G by counting spanning arborescences in H .

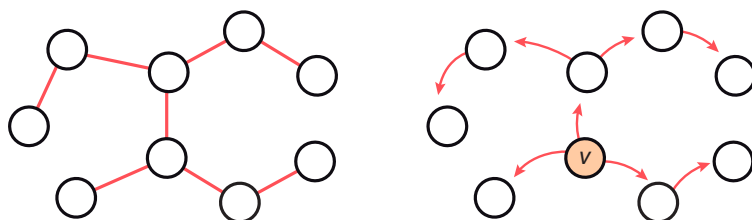


Figure 7.5: The undirected graph at left is a spanning tree for G in Figure 7.4, while the directed graph at right is a spanning arborescence for H (right side of Fig. 7.4) rooted at the shaded vertex v .

7.3 From Tutte to Kirchoff

The proofs of these theorems are long and so I will merely sketch some parts. One of these is the connection between Tutte’s directed Matrix-Tree theorem and Kirchoff’s undirected version. The key idea is illustrated in Figures 7.4 and 7.5. If we want to count spanning trees in an undirected graph $G(V, E)$ we should first make a directed graph $H(V, E')$ that has the same vertex set as G , but has two directed edges—one running in each direction—for each of the edges in G . That is, if G has an undirected edge $e = (a, b)$ then H has both the directed edges (a, b) and (b, a) .

Now we choose some arbitrary vertex v in H and count the spanning arborescences that have v as a root. It’s not hard to see that each spanning tree in G corresponds to a unique v -rooted arborescence in H , and vice-versa. More formally, there is a bijection between the set of spanning trees in G and v -rooted spanning arborescences in H : see Figure 7.5. The keen reader might wish to write out a careful statement of how this bijection acts (that is, which tree gets matched with which arborescence).

Finally, note that for our directed graph H , which includes the edges (a, b) and (b, a) whenever the original, undirected graph contains (a, b) , we have

$$\deg_{in}(v) = \deg_{out}(v) = \deg_G(v) \text{ for all } v \in V$$

where the in- and out-degrees are in H and $\deg_G(v)$ is in G . This means that the matrix L appearing in Tutte’s theorem is equal, element-by-element, to the graph Laplacian appearing in Kirchoff’s theorem. So if we use Tutte’s approach to compute

the number of spanning arborescences in H , the result will be the same numerically as if we'd used Kirchoff's theorem to count spanning trees in G .