Question	Learning Outcome	Solution
A1	**ILO6** – Construct the adjacency matrix of a graph and exploit the connection between powers of the adjacency matrix to count walks. Also, define the operations of *tropical arithmetic*, construct the weight matrix associated with a weighted graph and use its *tropical matrix powers* to find the lengths of shortest paths.	*The definition is bookwork, as is the construction of the adjacency matrix, but the last part of the question tests ILO6 at an intermediate level as the proof requested there is, though not difficult, unseen.*

(a)
- A vertex v in a digraph is *reachable* from another vertex u if there is a walk from u to v. Additionally, we say that a vertex is reachable from itself.

- A digraph $G(V, E)$ is *strongly connected* if it, for each pair of vertices $u, v \in V$, u is reachable from v and v is reachable from u.

(b) For a digraph $G(V, E)$ with vertex set $V = \{v_1, \ldots, v_n\}$, the *adjacency matrix* A is an $n \times n$ matrix whose entries are given by

$$A_{ij} = \begin{cases} 1 & \text{If } (v_i, v_j) \in E \\ 0 & \text{Otherwise} \end{cases}$$

For the matrix in the question, this means

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(c) The proposition is false and the graph from part (b) provides a suitable counterexample: it is clearly strongly connected, yet

$$A^\ell = \begin{cases} A & \text{If } \ell \text{ is odd} \\ I_2 & \text{If } \ell \text{ is even} \end{cases}$$

where I_2 is the 2×2 identity matrix. One can prove this in various ways, but I hope people will mention that A_{ij}^ℓ, the i, j-entry in A^ℓ, is the number of walks of length ℓ from v_i to v_j. In the graph at hand there are only two walks of length ℓ for each $\ell \geq 1$ and they differ only in where they start. Further, walks of even length finish on the vertex where they started, while those of odd length finish on the other vertex.

Question	Learning Outcome	Solution

A2   **ILO5** – Construct the *graph Laplacian* and apply the *Matrix-Tree Theorem* to count the number of spanning trees or spanning arborescences contained in a graph.

The question tests ILO5 at both basic and intermediate levels as construction of the graph Laplacan and the application of Tutte's Matrix-Tree theorem are bookwork, but the counting of spregs containing cycles requires an understanding of the key lemma in the proof of the theorem.

(a)
- A *spanning arborescence rooted at v* for a digraph $G(V, E)$ is a subgraph $T(V, E')$ with the properties that:
  - (i) it contains every vertex from $G(V, E)$;
  - (ii) every vertex is reachable from the root v;
  - (iii) if one ignores the directedness of the edges, the resulting graph $|T|$ is a tree.
- Spreg—short for "single predecessor graph"—is a term unique to this module. A *spreg with distinguished vertex v* is a digraph in which $\deg_{in}(u) = 1$ if $u \neq v$ and $\deg_{in}(v) = 0$.

(b) A theorem of Tutte says that the number of spanning arborescences in a digraph $G(V, E)$ can be computed as follows.

- Build a matrix L, the graph Laplacian, whose entries are given by

$$L_{i,j} = \begin{cases} \deg_{in}(v_j) & \text{If } i = j \\ -1 & \text{If } i \neq j \text{ and } (v_i, v_j) \in E \\ 0 & \text{Otherwise} \end{cases}$$

Here the relevant matrix is

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 2 \end{bmatrix}.$$

To count the spanning arborescences rooted at v_j, delete the j-th row and column of L to form $\hat{L}_j$. Here, as we want arborescences rooted at v_3, that's

$$\hat{L}_3 = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}.$$

The desired number of arborescences is then $\det(\hat{L}_3) = 6$.

(c) To specify a spreg with distinguished vertex v_3 one need only list the (single) predecessors of all the vertices other than v_3. The presence of the cycle fixes the predecessors of all the other vertices, so there is exactly one such spreg.

Question	Learning Outcome	Solution

A3 **ILO7** – Given a project defined by a set of tasks, along with their their durations and prerequisites, use *critical path analysis* to determine how quickly the project can be completed.

This is all bookwork: similar problems appear in examples sheets and on all available past papers. ILO7 tested at the level of competence.

(a) Here is a suitable graph.
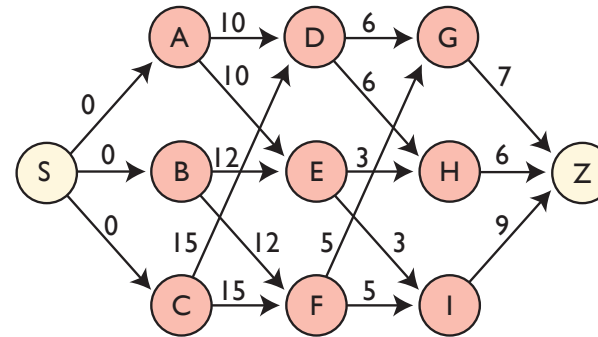
In addition to the vertices A–I representing the tasks there is a start vertex S and a finish vertex Z. The edge weights are the time required to complete the task at the tail vertex.

(b) In the graph below the vertices and edges on the critical path, $C - F - I$, are coloured green: they indicate that it will take a minimum of 29 days to complete the job. Those tasks that lie off the critical path are labelled with a pair $(t : T)$ where t is the earliest day on which the task could start and T the latest by which it must start if the project is to be completed in minimal time.
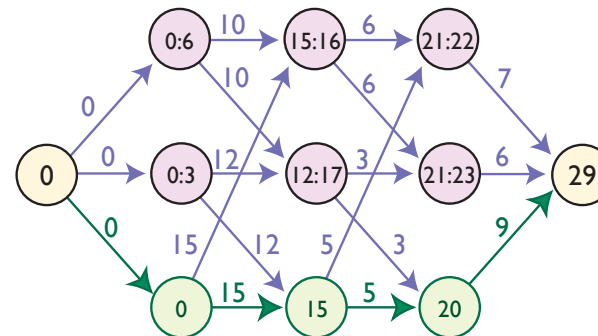
(c) The table, which one can read off the diagram above, is below

Task	A	B	C	D	E	F	G	H	I
Earliest possible start	0	0	0	15	12	15	21	21	20
Latest possible start	6	3	0	16	17	15	22	23	20

Question	Learning Outcome	Solution
B4	**ILO3** – Say what it means for a graph to be *Eulerian* and determine whether small graphs or multigraphs are Eulerian.	*The definitions are bookwork, as is the proof in part (b), but the reasoning required to support the answers in part (c) requires integration and deployment of key results from the sections of the course on Eulerian, Hamiltonian and planar graphs: ILO's 3, 4 and 8 are thus tested at a high level.*

ILO4 – Say what it means for a graph to be *Hamiltonian* and use the Bondy-Chávtal theorem to prove that a graph is Hamiltonian.

ILO8 – Say what it means for a graph to be *planar*; state and apply Kuratowski's theorem and determine whether a graph is planar or not.

(a)
- A graph $G(V, E)$ is *Hamiltonian* if it contains a cycle that includes every vertex in V.
- A graph $G(V, E)$ has an *Eulerian tour* or is *Eulerian* if it contains a closed trail that includes every edge in E.
- A *planar graph* $G(V, E)$ is one that has a *planar diagram*: a drawing in which the vertices are shown as points (or disks) and the edges are represented by line segments or arcs that connect the points (centres of disks) corresponding to the vertices on which the edge is incident. These arcs are allowed to intersect only at the vertices.
- The *girth* of a graph is the length of its shortest cycle. The girth is undefined for acyclic graphs.

(b) We are free to imagine that the underlying N-element set is $\{1, \ldots, N\}$ and to indicate its two-element subsets by $\{j, k\}$, where $j \neq k$ and $j, k \in \{1, \ldots, N\}$.

Now consider a particular vertex $\{j, k\}$. There are $(N - 2)$ elements in the underlying set that are different from both j and k and each such element gives rise to a pair of vertices adjacent to $\{j, k\}$. That is, for each i such that $i \neq j$ and $i \neq k$, we have two distinct adjacent vertices: one corresponding to the subset $\{i, j\}$ and another corresponding to $\{i, k\}$. Thus there are $2(N - 2) = 2N - 4$ distinct vertices adjacent to $\{j, k\}$ or, equivalently, $\deg(\{j, k\}) = 2N - 4$.

(c) Both the yes/no answer and the supporting argument are important here.

- The triangular graph T_6 is connected and every vertex has degree $12 - 4 = 8$, an even number. Hence T_6 is Eulerian.
- T_6 must be Hamiltonian. Let us refer to its vertex set as V. Then

$$|V| = \left(\begin{array}{c} 6 \\ 2 \end{array} \right) = \frac{6 \times 5}{2} = 15$$

and, as we argued above, each has degree 8. But then $\deg(v) \geq |V|/2$ for all $v \in V$ and so $[T_6]$, the closure of T_6, is isomorphic to K_{15} and the Bondy-Chvátal theorem implies that T_6 is Hamiltonian. Students could also invoke Dirac or Ore's Theorems, though we obtain those as corollaries of Bondy-Chvátal.

- The girth of T_6 is three, the smallest value it could possibly have. To see this, imagine that we use pairs drawn from the six-element set $\{1, \ldots, 6\}$ to label the vertices and specify the vertices as in part (b) above. Then the subgraph induced by $\{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$ consists of a cycle of length 3.

- In a planar graph on n vertices the number of edges m is bounded by

$$m \leq 3n - 6.$$

For T_6 the Handshaking Lemma says $m = (15 \times 8)/2 = 60$, while $3n - 6 = 3 \times 15 - 6 = 39$, so T_6 has far too many edges to be planar.

- As T_6 is nonplanar, we know from Kuratowski's Theorem that it contains a subgraph homeomorphic to K_5 or one homeomorphic to $K_{3,3}$. The question asks specifically about K_5. In fact, there are six subgraphs *isomorphic* (and hence automatically homeomorphic) to K_5 and the vertex set of any one of them makes clear what the others must be like. If we use the same notation as in the question about girth, the subgraph induced by the vertices

$$V' = \{\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}\}$$

is clearly isomorphic to K_5 as all these vertices share the element 1, so all are adjacent to each other.

| Question	Learning Outcome	Solution
B5	**ILO2** – Explain what the *chromatic number* of a graph is, determine it for small graphs and apply the idea to scheduling problems.	*The question tests ILO2 at both a basic level—as the reduction of problems about clashes to graph-colouring is well-explored in lecture and in the problem sets.*

(a)
- A *k-colouring* of a graph $G(V, E)$ is a map $\Phi : V \rightarrow \{1, \ldots, k\}$ with the property that $(u, v) \in E \Rightarrow \Phi(u) \neq \Phi(v)$.
- The *chromatic number* $\chi(G)$ of a graph G is the smallest k for which G has a k-colouring.

(b) The greedy colouring algorithm, which was discussed in lecture, will certainly be able to construct a $\Delta(G) + 1$ colouring. For when we come to choose a colour for some $v \in V$ it will have, at most, $\Delta(G)$ neighbours and so, with $\Delta(G) + 1$ colours available, there will always be an unused one that we can assign as $\Phi(v)$. And the existence of a $(\Delta(G) + 1)$-colouring implies that $\chi(G) \leq \Delta(G) + 1$. This bound is sharp, as every vertex in the complete graph K_n has degree $(n - 1)$, so $\Delta(K_n) = (n - 1)$. But then $\chi(K_n) = n = \Delta(K_n) + 1$. Another nice family of examples are the odd-length cycle graphs C_{2k+1}. They have

$$\chi(C_{2k+1}) = 3 = 2 + 1 = \Delta(C_{2k+1}) + 1.$$

(c) The idea is to construct and colour a graph whose vertices represent animals and whose edges connect animals that should not be housed together. The chromatic number of this graph is then the minimal number of enclosures and a corresponding colouring tells us how to house the animals: all the animals assigned the same colour go in the same enclosure. A suitable graph G is
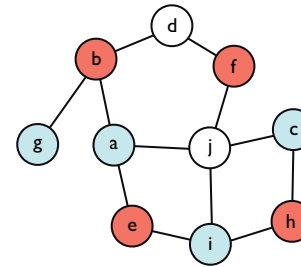
which has been coloured using three colours. On the one hand, this establishes that $\chi(G) \leq 3$, but on the other hand G contains a cycle of odd length, so we also know $\chi(G) \geq 3$. Thus $\chi(G) = 3$.

(d) The corresponding graph is isomorphic to the path graph P_3 which is illustrated below below and clearly has $\chi(P_3) = 2$.
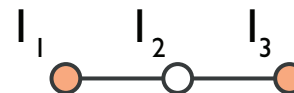
(e) Here the idea is to choose a set of intervals so that the corresponding graph is isomorphic to K_4. There are lots of ways to accomplish this, but one simple approach is to use nested intervals: $I_1 \subset I_2 \subset I_3 \subset I_4$. Thus, for example, the interval graph defined by $I_1 = [0, 1]$, $I_2 = [0, 2]$, $I_3 = [0, 3]$ and $I_4 = [0, 4]$ has chromatic number 4.

Question	Learning Outcome	Solution
B6	**ILO1** – Define what it means for two graphs to be isomorphic and determine, with rigorous supporting arguments, whether two (small) graphs are isomorphic.	*The definitions are bookwork and parts – are close to homework problems, though they're still difficult, so the question tests ILO1 at a high level.*

(a)
- The *degree sequence* of a graph $G(V, E)$ is a list of the degrees of G's vertices, arranged in ascending order.
- Two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are said to be *isomorphic* if there exists a bijection $\alpha : V_1 \to V_2$ with with property that given two vertices $a, b \in V_1$, the edge (a, b) is in E_1 if and only if $(\alpha(a), \alpha(b)) \in E_2$. In words: the two graphs are isomorphic if one can convert G_1 into G_2 by relabelling the vertices.
- A *tree* is a connected, acyclic graph.

(b) The statement is false, as is illustrated by the two graphs below. Both have degree sequence $\{1, 1, 2, 2, 2, 4\}$, but they're clearly non-isomorphic as the two vertices of degree 1 are both adjacent to the sole vertex of degree four in the graph on the left, but not in the one at right.
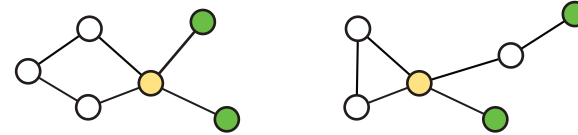
(c) This statement is true. It's a special case of the Handshaking Lemma, which says that for an arbitrary graph $G(V, E)$

$$\sum_{v \in V} \deg(v) \;=\; \sum_{j=1}^{n} d_j \;=\; 2|E|. \tag{1}$$

For a tree, we've proved in lecture that $|E| = |V| - 1$.

(d) This is also true, and is also a consequence of the Handshaking Lemma and the fact that $|E| = |V| - 1$ for a tree. Say that the tree has k leaves or equivalently, that it has k vertices of degree 1. Then the remaining vertices have degree 2 or more and we can write

$$\sum_{j=1}^{n} d_j \;\geq\; k + 2\left(|V| - k\right) \;=\; 2n - k.$$

On the other hand, we have (1) with $|E| = |V| - 1 = n - 1$ so

$$2|E| \;=\; 2n - 2 \;\geq\; 2n - k \qquad \text{or} \qquad k \geq 2, \tag{2}$$

which establishes that $d_1 = d_2 = 1$.

To see that $d_n > 1$, suppose otherwise—suppose $d_n = 1$. Then, as the d_j are arranged in ascending order, it must be true that $d_j = 1$ for all $1 \leq j \leq n$ and so

$$\sum_{j=1}^{n} d_j = n.$$

But this is incompatible with the Handshaking Lemma, for when $n \geq 3$, it's easy to see that $n < 2(n - 1)$.

(e) This is also true, as one can prove by induction on the number of entries in $\mathcal{D}$. Take $n = 2$ as the base case. Then $\sum_{j=1}^{n} d_j = 2n - 2 = 2$, which implies that $d_1 = d_2 = 1$. And this is certainly compatible with $\{1, 1\}$, the degree sequence of K_2, the only two-vertex tree.

Now suppose the result is true for all positive, non-decreasing sequences of length $2 \le n \le n_0$ that satisfy $\sum_{j=1}^{n} d_j = 2(n-1)$. Consider a sequence

$$\mathcal{D} = \{d_1, d_2, \ldots, d_{n_0+1}\}.$$

with the property that

$$\sum_{j=1}^{n_0+1} d_j = 2((n_0+1) - 1) = 2n_0.$$

Reasoning essentially identical to that in part (d) above establishes that $d_1 = d_2 = 1$ and $d_{n_0+1} > 1$. This in turn implies that somewhere in the sequence d_j jumps from 1 to some larger value. That is, there is a k in the range $2 < k \le n_0 + 1$ such that $d_{k-1} = 1$, but $d_k > 1$. Form a new, shorter sequence

$$\mathcal{D}' = \{d_1', \ldots, d_{n_0}'\} = \{d_2, \ldots d_{k-1}, (d_k - 1), d_{k+1}, \ldots, d_{n_0+1}\}.$$

$\mathcal{D}'$ has only n_0 entries, is arranged in ascending order and has the property

$$\begin{aligned}
\sum_{j=1}^{n_0} d_j' &= d_2 + d_3 + \cdots + (d_k - 1) + \cdots + d_{n_0+1} \\
&= \left(\sum_{j=1}^{n_0+1} d_j \right) - d_1 - 1 \\
&= 2n_0 - 2 \\
&= 2(n_0 - 1).
\end{aligned}$$

Hence the inductive hypothesis tells us that there is a tree whose degree sequence is $\mathcal{D}'$. Further, this tree has a vertex—call it v_k—whose degree is $(d_k - 1)$. And if we attach a single leaf to this vertex we increase its degree by one and create a tree whose degree sequence is $\mathcal{D}$.