Feedback About 2020's End-of-Term Formative Assessment
MATH20902: Discrete Mathematics

General remarks

- This assessment was not an exam, so there are no numerical marks to summarise. In any case, only 13 students submitted work, so there is not a lot to report.
- The notes about individual problems easier to follow if you have a copy of the assignment.

Remarks about individual problems

A1 People did well on this question, which is unsurprising as it appeared on 2017's exam and most of its solution is implicit in the Feedback.

(a) Most students were able to give correct definitions for the terms *reachable* and *strongly connected* (both appear in the lecture titled Walks, Trails, Paths and Connectedness), but one or two seemed not to realise that these terms are applicable only to directed graphs and so wrote, incorrectly, about "adjacent" vertices. There was also some imprecision in writing that spoiled definitions.

(b) Almost everyone wrote down the correct adjacency matrix,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

for the graph that appears in the question. The few mistakes that I saw involved incomplete explanations about how to construct the adjacency matrix of an arbitrary graph. This material appears in the lecture titled Representation, Sameness and Parts.

(c) Most students realised that the graph in part (b) provides a counterexample to the claim here. The key observation is that if A is the adjacency matrix constructed in part (b), then

$$A^\ell = \begin{cases} A & \text{If } \ell \text{ is odd} \\ I_2 & \text{If } \ell \text{ is even} \end{cases}$$

where I_2 is the 2×2 identity matrix. A few students merely computed a couple of examples of matrix-powers—typically A^2 and A^3—which does not provide a proof that A^ℓ contains zeroes for *all* $\ell \in \mathbb{N}$. Ideally one would prove this by induction, though none of the papers I read did this.

A2 This problem came from 2018's exam.

(a) As in question A1, most students gave good definitions here. Such shortcomings as there were came from:

- neglecting to explain key terms such as *rooted* or *spanning*;
- giving definitions that were too terse. So, for example, it was not enough to say

  > A *spanning arborescence* in a directed graph $G(V, E)$ is an arborescence that includes all the vertices in V.

  without explaining what an arborescence is. Similarly, it's not enough to say

  > A directed graph $T(V, E)$ is an *arborescence* if T contains a root and the graph $|T|$ that one obtains by ignoring the directedness of the edges is a tree.

without saying what a *root* is. On the other hand, you wouldn't need to define *tree*.

The principle here is that if a group of definitions appear in quick succession (those for *root*, *arborescence* and *spanning arborescence* are Definitions 7.6–7.8 and all appear one after the other in the printed lecture notes), then you need to spell out any and all of them that you use. So, for example, a good definition could be something like:

> An *arborescence rooted at v* is a directed graph $T(V, E)$ in which all other vertices are reachable from v and which becomes a tree when one ignores the directedness of the edges.

This definition doesn't use the term *root*, but instead explains that the vertex v must have the properties of a root. On the other hand, it does use the terms *reachable* and *tree* without further explanation, but that's okay as both were defined in earlier lectures.

(b) One can count spanning arborescences either with Tutte's Matrix Tree Theorem (and it's good to explain that this is what you're doing) or by drawing all associated spregs and then counting those that are spanning arborescences. The Matrix-Tree strategy was both more common and more generally successful, though there were a few small numerical mistakes in the construction of L and the evaluation of $\det(\hat{L}_3)$.

(c) Here I hoped students would either use the result from Proposition 9.4, which is about spregs that contain a single directed cycle, or use the approach from lecture to enumerate the relevant spregs, noting that the presence of the cycle (v_1, v_4, v_2, v_1) fixes the (single) predecessors of all those vertices that have one, so there can be only one such spreg. Most did this, but a few wrote cryptic, incorrect numerical answers with no supporting argument.

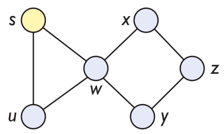
A3 Problems such as this one are standard parts of Discrete Maths exams, so people seem to have studied the topic carefully: all the answers I saw were largely correct, with only a few small mistakes, particularly in the computation of the latest possible starts. The correct times are listed below.

Task	A	B	C	D	E	F	G	H	I
Earliest possible start	0	0	0	15	12	15	21	21	20
Latest possible start	6	3	0	16	17	15	22	23	20

B4 A problem similar to this appeared on 2018's exam.

(a) The rather long list of definitions went well, though I saw a few small mistakes such as:

- saying that an an Eulerian tour is a cycle or a closed path. While this is possible, it isn't always true, as is illustrated in the graph below. The closed trail specified by the vertex sequence (s, w, x, z, y, w, u, s) is an Eulerian tour (traverses every edge exactly once and starts and finishes at the same vertex), but it is not a cycle.



- Some students neglected to mention planar diagrams in their definition of a planar graph, or when they did mention them, didn't describe them accurately.

Also, I found it odd that same students wrote

> A graph is *planar* if it contains a planar diagram.

Strictly speaking, this is wrong: graphs don't "contain" diagrams, they "have" them.

(b) Students did well on this, which is unsurprising as the same question appears as part of Problem 8 in Problem Set 1, for which complete solutions are available.

(c) The main issues were as follows:

Eulerian? Most people observed that all the vertices in T_6 have even degree, but a correct answer also needed to say that T_6 is connected to establish that it is Eulerian.

Hamiltonian? Many students (correctly) invoked one of Dirac's, Ore's or the Bondy-Chvátal Theorem, but a few forgot to mention $|V| = n = 15$, an important ingredient in all these arguments. One student constructed an explicit Hamiltonian tour.

girth? Most people chose the simplest approach to this problem and gave an example of a cycle of length 3, which is the shortest possible cycle in an undirected graph.

planar? Almost all students answered correctly, saying that T_6 can't planar as it has far too many edges, but not as many as I had hoped gave good supporting arguments. One should use the inequalities proved over the last two weeks of lectures. The relevant one here says that if $G(V, E)$ is planar, then $|E| \leq 3|V| - 6$.

contains K_5? A few people tried to use Kuratowski's theorem here, but that's not so helpful as it only tells us that T_6 must contain $K_{3,3}$ or K_5 homeomorphically. In fact T_6 contains six subgraphs *isomorphic* to K_5 and the vertex set of any one of them makes it clear what the others must be like. If we take the underlying six-element set to be $\{a, b, c, d, e, f\}$ and write its two-element subsets as pairs of letters then the vertices

$$\{ab, ac, ad, ae, af\}$$

are all connected to each other (all share an a) and so clearly induce a subgraph isomorphic to K_5.

B5 This too is an old exam problem, though I last used it in 2015.

(a) The definitions went well.

(b) This also went well, unsurprisingly, as it is part of Problem 4 on Problem Set 2.

(c) Most people gave reasonably good answers here too, perhaps again becasue the problem is essentially the same as Problem 2 on Problem Set 2. Most of the answers I saw would have been improved by an explanation of how to reduce the problem about the minimal number of enclosures to one about graph colouring.

(d) This problem was easy for anyone who had done this year's coursework.

(e) This problem, too, was easy for those who had done the coursework.

B6 This problem appeared on 2016's exam.

(a) The definitions went well, which was especially cheering as previous students have often made mistakes with the definition of *isomorphic*.

(b) Everyone agreed that the statement here was false and found the only counterexample. The proofs of non-isomorphism, all done by contradiction, were also generally clear and correct.

(c) Most students recognised that that is just an application of the Handshaking Lemma to trees, where we know $|E| = |V| - 1$, but a few offered much more elaborate arguments.

(d) I saw several clear, correct arguments here, but also a number of confused ones. Both $d_1 = d_2 = 1$ and $d_n > 1$ follow readily from the Handshaking Lemma in the form used in part (c). Also, the chapter of the lecture notes entitled Trees & Forests includes a full proof for a result, Lemma 6.1, that is equivalent to $d_1 = d_2 = 1$.

(e) This is a somewhat harder question and few students gave wholly satisfactory proofs, even though the necessary arguments appear in the solution to Problem 7 from Problem Set 3.