***Feedback About 2018's Exam in***
**MATH20902: Discrete Mathematics**

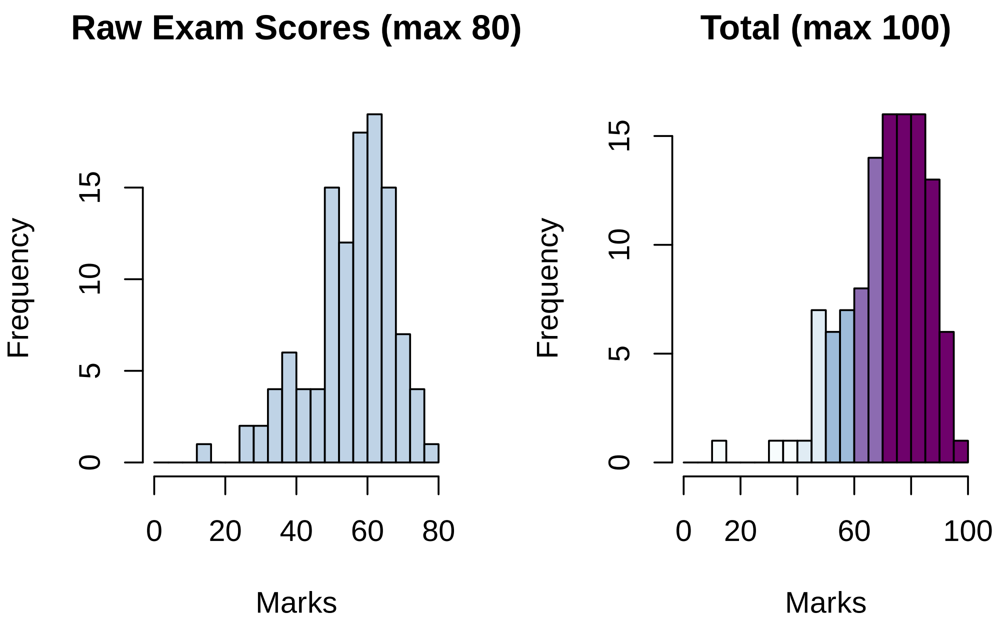



Figure 1: *Histograms of the raw exam results (left panel) and final marks (right panel). The histogram of total marks (where "total" means exam plus coursework) has been shaded to indicate degree classes. These plots include data for all 114 students who sat the exam.*

## General remarks

- The final marks summarised in the panel at right above have not been adjusted to take account of Mitigating Circumstances nor have they been moderated (that is, scaled) to make them more comparable to other exams in the School.
- Many people did very well on the exam: although there were no perfect papers, 27 students had exam marks of 65 or better (out of 80). Here is a summary of the final marks, by degree class.

| **Result:** | First | 2(i) | 2(ii) | Third | Fail |
|---|---|---|---|---|---|
| **Range:** | 70–100 | 60–69 | 50–59 | 40–49 | 0–39 |
| **Number of students:** | 68 | 22 | 13 | 8 | 3 |
| **Fraction of students:** | 59.6% | 19.3% | 11.4% | 7% | 2.7% |

- Just over a fifth of all students attempted all three of the part B problems. For these students only the best two part B scores contributed to the total exam mark.

| **Part B problems attempted:** | B4 & B5 | B4 & B6 | B5 & B6 | All |
|---|---|---|---|---|
| **Number of students:** | 57 | 24 | 10 | 23 |

- These notes about individual problems easier to follow if you have a copy of the exam.

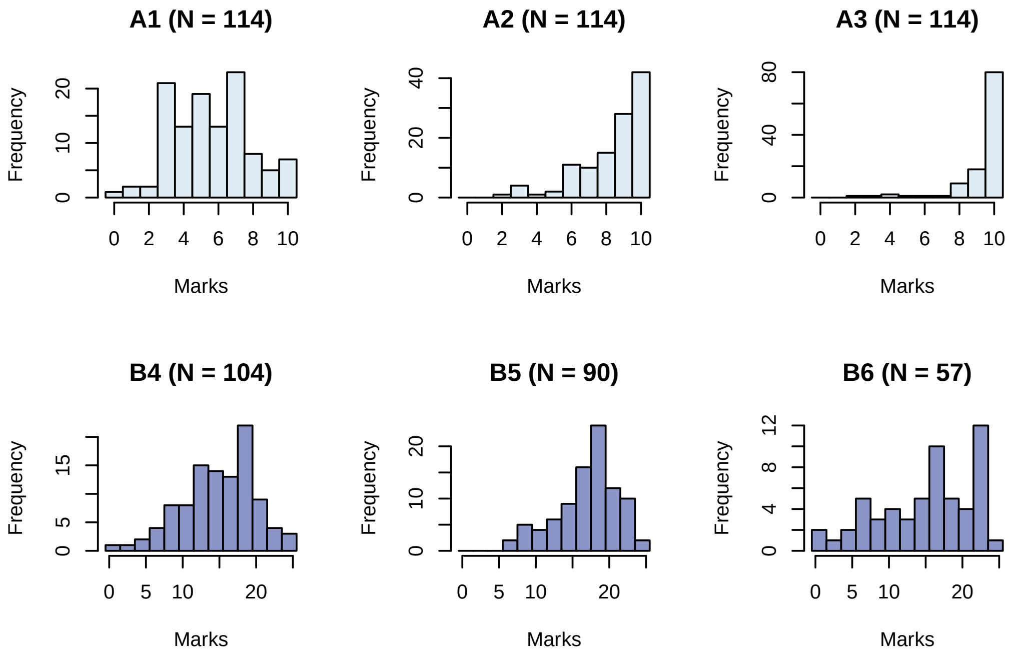



Figure 2: *Histograms for the individual questions: note that the vertical scale varies from question to question. Those for the required Part A questions (10 marks apiece) are plotted light blue, while those for the Part B questions (best two out of three, 25 marks apiece) are slightly darker. In the latter group, some of the very lowest marks come from students who attempted all three part B questions and so these scores didn't usually contribute to the total exam mark.*

## Remarks about individual problems

**A1** Almost all students got the 3 marks associated with the definitions, but the rest of the problem proved somewhat harder than I'd hoped: although there were 7 perfect answers, the average mark for this question was only around 5.6/10. The question asked you to prove two things:

- If there is a saturated hydrocarbon with formula $\mathrm{C}_m\mathrm{H}_n$, then $n = 2m + 2$.
- If $m$ is a positive integer and $n = 2m + 2$, then there exists an example of a saturated hydrocarbon with formula $\mathrm{C}_m\mathrm{H}_n$.

The first of these is fairly easy to establish using the Handshaking Lemma and the observation that a saturated hydrocarbon is—if viewed as a graph—a tree with hydrogen atoms as leaves and carbon atoms as internal nodes with degree 4. If you used the Handshaking Lemma here I wanted to see it mentioned by name explicitly, rather than just used as an intermediate step in a wordless sequence of formulae.

The second thing is most easily proven by constructing a family of examples. One especially simple family has the $m$ carbon atoms arranged in a linear chain, like the vertices of a path graph, with the $2m+1$ hydrogens attached to the carbons as in the example at left in Figure 3.

Many students tried to make proofs by induction for one or both of the statements above. Some of those for the second statement were successful, but most of those for the first (saturated hydrocarbon $\Rightarrow n = 2m + 1$) were not, mainly because they assumed, without supporting argument, that *all* saturated hydrocarbons can be built sequentially, by repeatedly adding a single carbon to a smaller saturated hydrocarbon. This is true, but needs an

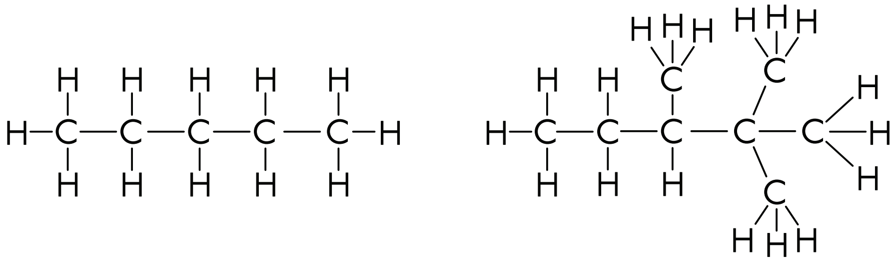


Figure 3: *Two saturated hydrocarbons: Problem A1 turns on the observation that such a molecule is—when viewed as a graph—a tree.*

inductive proof that starts with a saturated hydrocarbon having, say, $n_0 + 1$ carbons and *removes* a carbon to get a smaller saturated hydrocarbon.

A few students found another way to approach the problem by using a result from the Problem Sets that says that if one has a non-decreasing sequence of positive integers $\mathcal{D} = (d_1, \ldots, d_N)$ with the property that

$$\sum_{j=1}^{N} d_j = 2(N-1),$$

then there exists a tree whose degree sequence is $\mathcal{D}$.

Finally, a few students became muddled up because the problem involved the symbols $n$ and $m$ for, respectively, the numbers of hydrogen and carbon atoms. Thus, although we have often used $n$ for the number of vertices in a graph and $m$ for the number of edges, the graphs associated with a hydrocarbon whose chemical formula is $\mathrm{C}_m\mathrm{H}_n$ have $|V| = n + m = 3m + 2$ vertices and $|E| = (4m + n)/2 = 3m + 1$ edges.

**A2** Perhaps because similar questions have appeared on many recent exams, this proved an easy question: the average was 8.4/10.

(a) As in question A1, most students got most of the marks available for definitions. Those who lost marks made two main kinds of mistakes.

- Some neglected to explain key words such as *rooted* or *spanning*.
- Some gave definitions that were too terse. So, for example, it was not enough to say

  > A *spanning arborescence* in a directed graph $G(V, E)$ is an arborescence that includes all the vertices in $V$.

  without explaining what an arborescence is. Similarly, it's not enough to say

  > A directed graph $T(V, E)$ is an *arborescence* if $T$ contains a root and the graph $|T|$ that one obtains by ignoring the directedness of the edges is a tree.

  without saying what a *root* is. On the other hand, you wouldn't need to define *tree.*

  The principle here is that if a group of definitions appear in quick succession (those for *root*, *arborescence* and *spanning arborescence* are Definitions 7.6–7.8 and all appear one after the other in the printed lecture notes), then you need to spell out any and all of them that you use. So, for example, I awarded full marks to students who wrote something like

  > An *arborescence rooted at* $v$ is a directed graph $T(V, E)$ in which all other vertices are reachable from $v$ and which becomes a tree when one ignores the directedness of the edges.

This definition doesn't use the term *root*, but instead explains that the vertex $v$ must have the properties of a root. On the other hand, it does use the terms *reachable* and *tree* without further explanation, but that's okay as both were defined in earlier lectures.

(b) One can count spanning arborescences either with Tutte's Matrix Tree Theorem (and it's good to explain that this is what you're doing, though I didn't deduct any marks for correct answers that didn't mention Tutte) or by drawing all associated spregs and then counting those that contain no cycles: both strategies were acceptable and received full marks when correct. The Matrix-Tree strategy was both more common and more generally successful, though there were a lot of small mistakes in the construction of $L$, mainly involving the entries $L_{2,2}$ and $L_{4,2}$, which should be 3 and -1, respectively, and in the evaluation of $\det(\hat{L}_2)$, which should be 4.

(c) This part proved even easier than the previous part, with the best answers looking something like

The total number of spregs with distinguished vertex $v$ is given by

$$\prod_{u \neq v} \deg_{in}(u),$$

so for $v_2$ in the the graph from the exam, there are

$$\deg_{in}(v_1) \times \deg_{in}(v_3) \times \deg_{in}(v_4) \;=\; 2 \times 2 \times 2 \;=\; 8$$

spregs (see Figure 4 for sketches).

The most common mistakes in this part included:

- the surprisingly common claim

$$\prod_{u \neq v_2} \deg_{in}(u) \;=\; \deg_{in}(v_1) \times \deg_{in}(v_3) \times \deg_{in}(v_4) \;=\; 2 \times 2 \times 2 \overset{??}{=} 6;$$

- drawing spregs where some vertex other than $v_2$ had no predecessors;
- neglecting to sketch the spregs.

**A3** As I'd hoped, most students found this question easy. The average was around 9.3/10 and 80 students got 10/10. The main ways people lost marks were by:

- in part (a), representing the problem with some graph other than the sort used in lecture and not explaining how this non-standard graph worked;
- also in part (a), drawing an undirected graph, even though the problem asked explicitly for a directed one (one mark deducted) or drawing a graph with no edge weights and no other indication of the times required for the various tasks;
- failing to indicate the critical paths in part (b). There were two, $S$-$A$-$B$-$C$-$G$-$Z$ and $S$-$A$-$D$-$F$-$G$-$Z$, leading to a minimum time-to-completion of 21 days. A few students found this result by exhaustive enumeration of all paths and received full credit.
- As all vertices except $E$ lie on one of the critical paths, part (c) was pretty easy: the only way students lost marks here was by not answering the question (that is, not giving earliest and latest starts for each vertex).

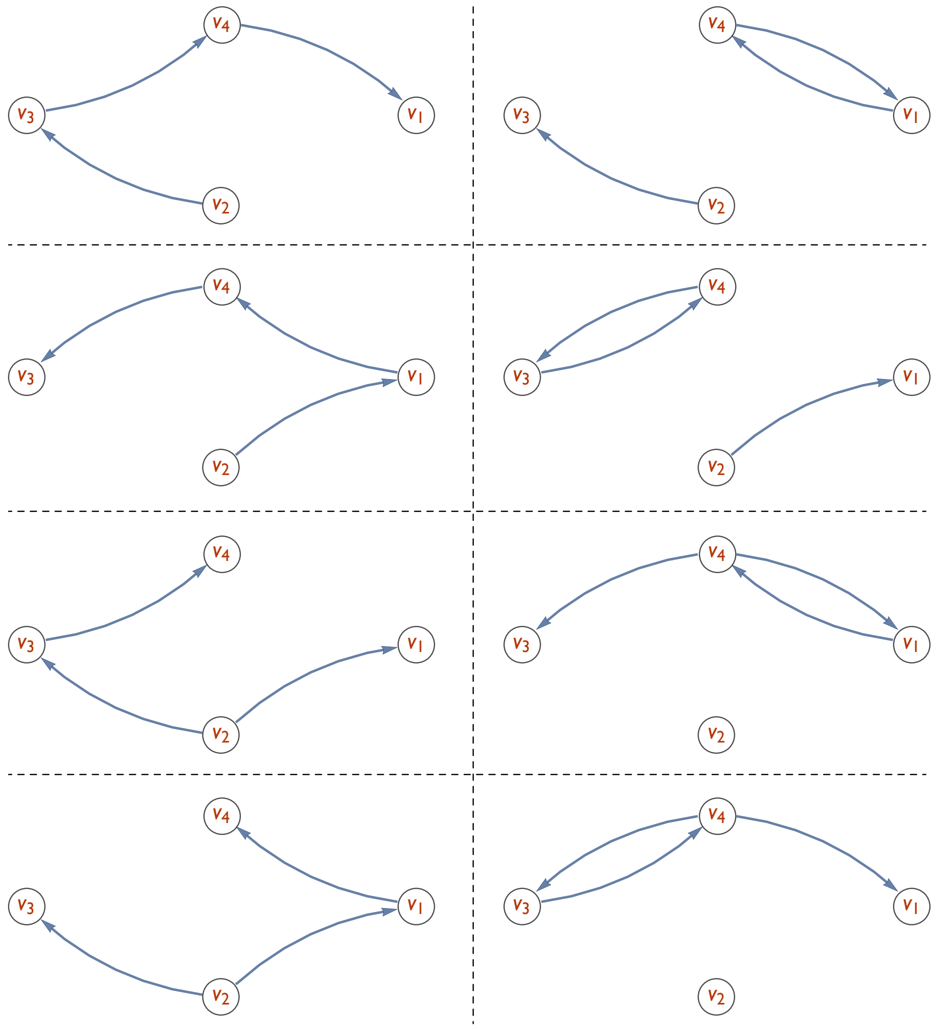



Figure 4: *The eight spregs with distinguished vertex $v_2$ present in the graph from question A2. The four in the left column are spanning arborescences rooted at $v_2$, while while those in the right column contain cycles.*

**B4** This was the most popular of the Part B problems: 104 out of 114 students tried it. Answers varied wildly in quality, from 16 scripts with a score less than 10 on up to 10 scripts with scores in excess of 20/25: the average was 14.9/25.

(a) In contrast to Part A, where most students gave correct definitions and got all the associated marks, here there were more mistakes. Examples include:

- forgetting to mention that $E \neq \emptyset$ in a bipartite graph;
- a definition of $k$-colouring that claimed the associated mapping $\phi : V \to \{1, \ldots, k\}$ is a bijection. This is possible—for example for an $n$-colouring of $K_n$—but it's not true in general.
- A claim, in the definition of a $k$-colouring, that "$\phi(u) \neq \phi(v)$ if and only if $(u, v) \in E$". The right thing to say is that $(u, v) \in E \Rightarrow \phi(u) \neq \phi(v)$.

(b) A substantial minority of students didn't seem to understand what it takes to prove an "if and only if" result. A complete answer to the question requires proofs of two statements:

- If a graph $G$ is bipartite, then $\chi(G) = 2$.
- If $\chi(G) = 2$ then $G$ is bipartite.

Many students only proved the first of these.

(c) Many students skipped this, though I did see a few examples of a lovely proof by contradiction based on the result form part (b). Suppose for contradiction a bipartite graph $G(V, E)$ contains a cycle of odd length and call the subgraph consisting of this cycle $C$. On the one hand we know that $\chi(C) = 3$, because $C$ is a cycle of odd length. But on the we also know that because $C$ is a subgraph of $G$, $\chi(C) \leq \chi(G) = 2$.

(d) Nearly everyone found a 2-colouring for $H$ and so established $\chi(H) \leq 2$, then completed the proof that $\chi(H) = 2$ by observing that $H$ contains edges, so can't have a 1-colouring, which means $\chi(H) \geq 2$. One student claimed that any graph in which all the cycles have even length must be bipartite, but while this is true, it's nontrivial to prove and so I didn't award many marks for the bare assertion.

A surprising number of students wrote something like "$u$ maps to $v$" to mean "the edge $(u, v)$ is present in the graph." This was odd, but not exactly wrong and so I didn't deduct any marks for it.

(e) Most students said, correctly, that $H$ is not Hamiltonian, but fewer offered rigorous justifications. The best answers pointed out that $H$ has 11 vertices and so any Hamiltonian cycle would have length 11, an impossibility in a bipartite graph. Common mistakes included:

- confusion between criteria for Hamiltonian cycles and Eulerian ones. These students usually mentioned the presence of a vertex of odd degree, but that's not relevant here.
- Misapplication of Dirac and/or Ore's Theorems. These results are of the general form

  *If all the vertices in $G$ have high degree, then $G$ is Hamiltonian.*

  Some students tried to use these result "backwards", claiming that as all the vertices in $H$ have low degree, (for example, $\deg(v) < |V|/2$ for all $v \in V$), then $H$ can't be Hamiltonian. But this is a mistake in logic: Dirac's and Ore's Theorems can only be used to prove a graph *is* Hamiltonian, not to prove that it isn't. After all, the cycle graphs are all Hamiltonian—indeed, a cycle graph is nothing but one big Hamiltonian cycle—yet all their vertices have degree 2.

**B5** This was the second most popular of the Part B problems: 90 out of 114 students attempted it and the average was 17.1/25.

(a) The definitions went well for the most part. Most of the (few) mistakes came from:

- defining a *walk* as a sequence of edges $(e_1, \ldots, e_\ell)$ without mentioning that these edges need to join up in the sense that there should be corresponding sequence of vertices $(v_0, \ldots, v_\ell)$ such that $e_j = (v_{j-1}, v_j)$;
- mixing up the adjacency matrix and the graph Laplacian or the diagonal matrix $D$ in which $D_{jj} = \deg(v_j)$.

(b) I had hoped this would be easy, as it is similar to problems from Foundations of Pure Maths, but a surprising number of students seem to have had trouble recalling what an equivalence relation is and how to construct equivalence classes. Typical problems included:

- forgetting to mention that "is strongly connected to" is reflexive by definition, in that we always say that a vertex is strongly connected to itself;
- forgetting the equivalence class that consists of $v_4$ on its own. The other equivalence classes/strong components were $\{v_1, v_2, v_3\}$ and $\{v_5, v_6, v_7\}$.
- Many students listed pairs of strongly connected vertices, but such pairs are not strong components, which is what the question requested.

(c) Most students did well here though a few lost a mark for constructing the transpose of the adjacency matrix rather than $A$ itself.

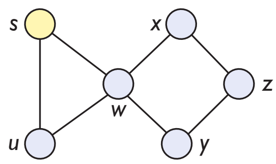
(d) Here again, people did well, though for full credit I wanted to see some discussion of $a \oplus \infty$ and $a \otimes \infty$.

(e) There was a small subtlety in this question in that the matrix $W$ here has $\infty$ in the diagonal elements, as opposed to the 0's that appeared in the matrix in Lecture 14. This means that, for the matrix in the exam, $[W^{\otimes \ell}]_{ij}$ gives the length of a shortest walk of length $\ell$ from $v_i$ to $v_j$. Many students side-stepped this issue by simply computing $W^{\otimes \ell}$ directly, sometimes using repeated squaring

**B6** This was the least popular question in Part B, with only 57/114 attempts. The average was only 12.2/25, but this was pulled down by a number of students who did part (a), but nothing more. Of those who attempted the whole problem, 19 got a mark of 20 or better.

(a) The rather long list of definitions went well, though some students lost a mark or two in one of the following ways:

- saying that an an Eulerian tour is a cycle or a closed path. While this is possible, it isn't always true, as is illustrated in the graph below. The closed trail specified by the vertex sequence $(s, w, x, z, y, w, u, s)$ is an Eulerian tour (traverses every edge exactly once and starts and finishes at the same vertex), but it is not a cycle.




- Some students neglected to mention planar diagrams in their definition of a planar graph.

Also, I found it odd that many students wrote

> A graph is *planar* if it contains a planar diagram.

Strictly speaking, this is wrong: graphs don't "contain" diagrams, they "have" them, but I was more concerned that people should give a complete definition, including one for *planar diagram*, and so ignored this minor mistake.

(b) Students did surprisingly poorly on this problem, given that it was among those from the Problem Sets that I suggested you study.

(c) I worried that, despite my warning in the question, time-pressured students would answer this part of the question for $T_4$, which was pictured in the exam, rather than $T_6$, which is what the question was about. Happily this didn't happen very often. The main issues were as follows:

**Eulerian?** Most people observed that all the vertices in $T_6$ have even degree, but a correct answer also needed to say that $T_6$ is connected to establish that it is Eulerian.

**Hamiltonian?** Many students (correctly) invoked one of Dirac's, Ore's or the Bondy-Chvátal Theorem, but a few forgot to mention $|V| = n = 15$, an important ingredient in all these arguments. A handful of students tried to construct an explicit Hamiltonian tour, but as such a tour has length 16, this was both hard to do and hard to check.

**girth?** The simplest way to do this was to give an example of a cycle of length 3 and then say that in an undirected graph, all cycles have length 3 or more. A few students mentioned, correctly, that $T_3$ is a subgraph of $T_6$ and I accepted this.

**planar?** Essentially all students who got this far answered correctly, demonstrating that $T_6$ can't planar as it has far too many edges.

**contains $K_5$?** A few people tried to use Kuratowski's theorem here, but it's not helpful as it involves subgraphs *homeomorphic* to $K_5$, while the question concerns subgraphs *isomorphic* to $K_5$. In fact $T_6$ contains six subgraphs isomorphic to $K_5$ and the vertex set of any one of them makes it clear what the others must be like. If we take the underlying six-element set to be $\{a, b, c, d, e, f\}$ and write its two-element subsets as pairs of letters then the vertices

$$\{ab, ac, ad, ae, af\}$$

are all connected to each other (all share an $a$) and so clearly induce a subgraph isomorphic to $K_5$.