Feedback About 2020's Coursework in
MATH20902: Discrete Mathematics

I made the notes below while marking the coursework. I also added comments to individual papers and would be happy to discuss these. The marked work is available through Blackboard.

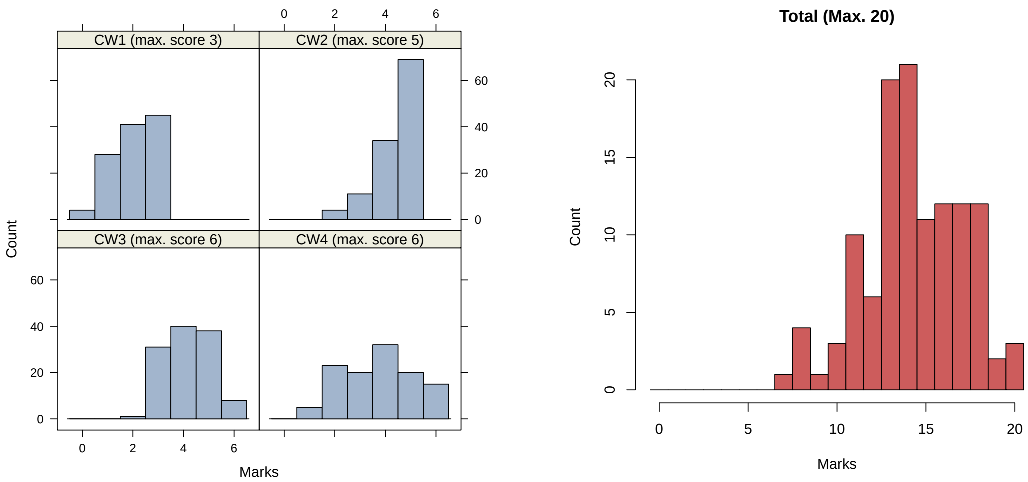

Figure 1: *Histograms for the individual questions and for the total score (right). The histograms above include marks for 118 students.*

Overall Remarks

- Generally speaking, people did well: the average for the whole assignment was around 14.3/20 or, approximately 71.5%.
- I saw a lot of good arguments and examples while marking these scripts. There were, however, some odd and surprisingly common misconceptions: nothing in the definition of an interval graph excludes graphs with multiple connected components.
- Quite a few people seem to have looked at other sources and to have documented this properly: I saw bibliographic references to books, articles and sets of lecture notes from other universities. Where ideas from outside the course appeared without appropriate referencing, I deducted a mark.
- I marked the written component papers with an eye to good mathematical style (see the question-by-question notes below for details). To get full marks you needed to make an argument that was technically correct, rigorous, clear and concise.

Remarks about individual problems

(1) (Constructing interval graphs [3 marks]).
I hoped this would be an easy question and it seems to have been: all 118 students answered it and the average was just under 2.1 marks, or around 70%. To get all the marks here you needed to provide an *explicit* construction for the relevant intervals. For example, for K_n one might use $I_j = (-j, j)$. Many students lost marks by assuming that the intervals were closed, so that, for example, $\{(0,1),(1,2),(2,3)\}$ would be a suitable set of intervals for P_3, but the assignment specifically said to use *open* intervals. Others offered collections that included multiple copies of the same interval, but the definition of intervals graphs on the first page of the assignment makes it clear that the intervals should be distinct.

A substantial minority of students constructed collections of intervals that intersected as they should, but didn't have enough members. Typical mistakes of this kind included:

- a set of only $n-1$ intervals for P_n;
- a set of k intervals for the tree in part (c). The graph with k leaves should have $k+1$ vertices/intervals.

A small minority of students wrote down something like:

$$\text{Choose } \{I_1,\, I_2,\, \ldots, I_n \mid I_j \cap I_k \neq \emptyset \text{ for } 1 \leq j \leq k \leq n\}$$

for part (a), which is just a restatement of the definition of K_n in the language of interval graphs. But this isn't sufficient, as one needs to establish that such a collection actually *exists* and the easiest way to do that is to write down an explicit expression for a family of intervals I_j that have the required intersection properties. If you lost marks here and still don't understand why, ask yourself:

- how the correctness of your answer depends on properties of intervals;
- why your approach wouldn't also work as an explanation of how to choose intervals whose corresponding interval graph is isomorphic to C_4.

(2) (Two "Benzer problems" [5 marks]).
Everyone answered this fairly easy question and many got full marks: the average was around 4.4, or just over 88%. To get all the marks here you had to:

- draw the graphs associated with the matrices A_1 and A_2;
- determine whether these graphs could be represented by interval graphs;
- provide either a suitable collection of intervals, arranged in lexicographic order or an explanation of why no such collection exists. Thus the answer for A_1 needed to include something like

$$\begin{aligned}
I_6 &= (0, 1) \\
I_3 &= (0.5, 1.5) \\
I_8 &= (1, 2) \\
I_1 &= (3, 4) \\
I_7 &= (3.5, 4.5) \\
I_2 &= (5, 6) \\
I_5 &= (5.5, 6.5) \\
I_4 &= (7, 8)
\end{aligned}$$

  where the intervals are labelled to match the rows and columns of the matrix in the problem, but are arranged in lexicographic order.

When explaining why the matrix in part (b) doesn't give rise to an interval graph, many people wrote something like "G contains a cycle of length 4 as a subgraph", but this isn't really enough: one needs to say that this cycle is an *induced* subgraph.

(3) (**Berge's mystery [6 marks]**).
Everyone answered this too and the average was around 4.2 out of 6, or just ovunderer 70%. The idea here is that if the women are telling the truth, then we ought to be able to construct an interval graph whose vertices are labelled by the wives' names (or first initials) and whose edges connect women who met each other. We expect this to be an interval graph, with the intervals being the periods during which a given wife was at the Duke's castle.

Many students didn't describe the construction at all (which is poor mathematical style and cost marks) and those who did ranged from the uninformative

> $\cdots$ *the graph of this is:*

written just above a drawing to the much better

> *We can convert the problem into an integral graph one where the intervals are the periods the wives spent visiting* $\cdots$.

The graph based on the women's testimony is provably *not* an interval graph and so at least one woman is lying. At least two lines of reasoning suggest Alice is the culprit:

- the graph contains three induced subgraphs that are isomorphic to C_4 and her vertex appears in all of them;
- if you delete Alice's vertex, the graph that remains is an interval graph and hers is the only vertex with this property. Many students established only the weaker result that Alice's vertex is unique in that deleting it leaves a graph that has a perfect elimination scheme, but for full credit I wanted to see a set of intervals.

Quite a few students solved the rather harder problem of finding the minimal number of visits that Alice could have made: it's three.

(4) (**Perfect elimination scheme $\Longrightarrow$ interval graph? [6 marks]**).
This was the hardest problem: 3 students skipped it, though the average for those who did answer was around 3.7 out of 6, or 62%. The converse of proposition 2 is false: it's possible for a graph to have a perfect elimination scheme, yet not be an interval graph. For full marks I wanted to see a correct yes/no answer, a counterexample and an explanation for why the example couldn't be an interval graph.

A few students gave very good answers based on wider reading: a key idea here is that of a *chordal* graph. A graph is chordal if it does not contain any induced subgraphs isomorphic to C_n with $n \geq 4$. Equivalently, a graph$G(V, E)$ is chordal if every cycle appearing in G is either isomorphic to C_3 or has a *chord*, an edge connecting two vertices that aren't adjacent in the cycle. This definition allows one to state the following proposition, which is arguably the closest true statement to the converse of our Proposition 2:

Proposition. *If $G(V, E)$ has a perfect elimination scheme, then it is chordal.*

We know from the lemma in the assignment that interval graphs are chordal, so the issue in this question is whether there are any chordal graphs that aren't interval graphs. It turns out that there are, but there aren't so many. An early characterisation of interval graphs due to Lekkerkerker and Boland[1] allowed them to prove:

Theorem (Lekkerkerker and Boland (1962)). *A graph $G(V, E)$ is an interval graph if and only if it does not contain an induced subgraph isomorphic to one of those*[2] *in Figure 2.*

This set of forbidden subgraphs includes many that have a perfect elimination scheme and so provide the sort of counterexample we sought. My solution (see below) is based on R_2 and many correct student answers involved the same graph, though B_1 and L_1 were also pretty common.

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[1] C. Lekkerkerker and J. Boland (1962), Representation of a finite graph by a set of intervals on the real line, *Fundamenta Mathematicae*, **51**:45–64.

[2] The figure is adapted from one in W. T. Trotter (1992), *Combinatorics and partially ordered sets : dimension theory*, Johns Hopkins University Press, Baltimore.

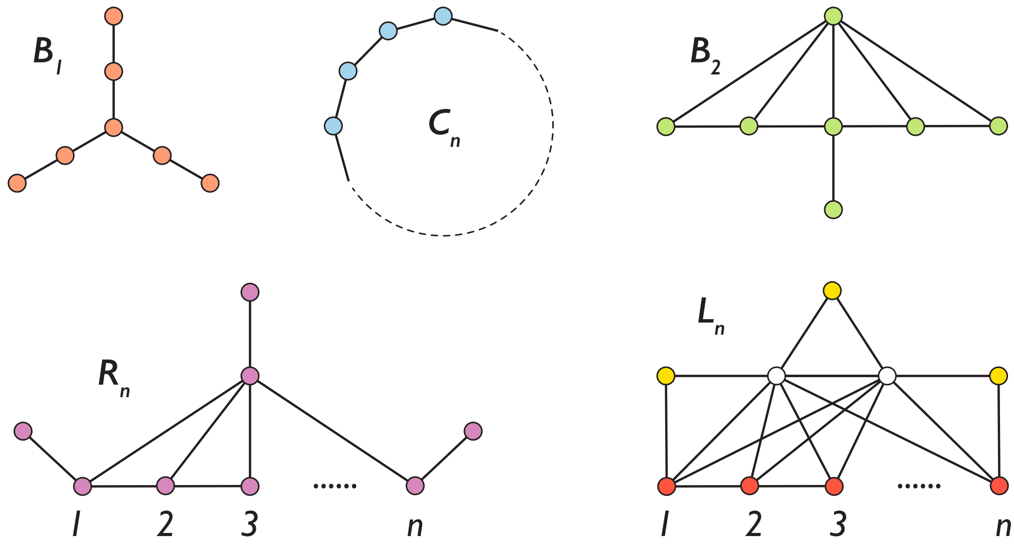

Figure 2: *Forbidden induced subgraphs for interval graphs. In addition to the two single graphs B_1 and B_2 there are three infinite families: C_n for $n \geq 4$; R_n with $n \geq 2$, which consists of a triangulated cycle of length $(n+1) \geq 3$ as well as three vertices of degree one, and L_n, which consists of a graph isomorphic to a complete bipartite graph $K_{2,n}$ with $n \geq 1$ (shown in red and white) as well as three other (yellow) vertices.*

Finally, I include a model answer, to give an idea of the level of detail and rigour required.

Proof. The converse of Proposition 2 is not true, as is illustrated by the counterexample in Figure 3. Call this graph $G(V, E)$ and note that $\{a, b, d, c, e, f\}$ is a perfect elimination scheme for G.

Suppose for contradiction that there is some collection of intervals $I_a = (l_a, r_a), \dots I_f = (l_f, r_f)$ whose interval graph is G. As the subgraph induced by $\{a, b, c, d\}$ is isomorphic to the path graph P_4, it must be true that either

$$l_b < r_a < l_c < r_b < l_d < r_c \qquad \text{or} \qquad l_c < r_d < l_b < r_c < l_a < r_b, \tag{1}$$

Both possibilities are illustrated in Figure 4.

Now consider I_e. The presence of the edges (b, e) and (c, e) imply that $I_b \cap I_e \neq \emptyset$ and $I_c \cap I_e \neq \emptyset$, but the absence of the edges (a, e) and (d, e) requires $I_a \cap I_e = I_d \cap I_e = \emptyset$. These observations, along with (1), imply that either

$$l_b < r_a < l_e < r_e < l_d < r_c \qquad \text{or} \qquad l_c < r_d < l_e < r_e < l_a < r_b. \tag{2}$$

Either way, we can conclude that

$$I_e \subset I_b \cup I_c. \tag{3}$$

Now, the presence of the edge (e, f) means that $I_e \cap I_f \neq \emptyset$, but this, along with (3), implies that

$$I_b \cap I_f \neq \emptyset \qquad \text{or} \qquad I_c \cap I_f \neq \emptyset$$

which would in turn imply the existence of at least one edge that is not present in the graph. Thus no collection of intervals can have G as its interval graph, even though G has a perfect elimination scheme. $\square$

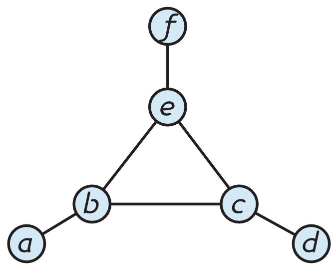

Figure 3: *A counterexample to the claim that all graphs that have a perfect elimination scheme are interval graphs.*

Figure 4: *Arrangements of intervals consistent with the observation that the vertices* $\{a, b, c, d\}$ *induce a subgraph isomorphic to* P_4*. The intervals* I_a *and* I_d *may also be proper subsets of* I_b *and* I_c*, respectively, but this does not affect the inequalities in Eqns.* (1) *or* (2).