Feedback About 2019's Coursework in
MATH20902: Discrete Mathematics

I made the notes below while marking the coursework. I also made remarks on individual papers and would be happy to discuss these. The marked work is available at the Reception desk near the entrance to the Alan Turing Building.

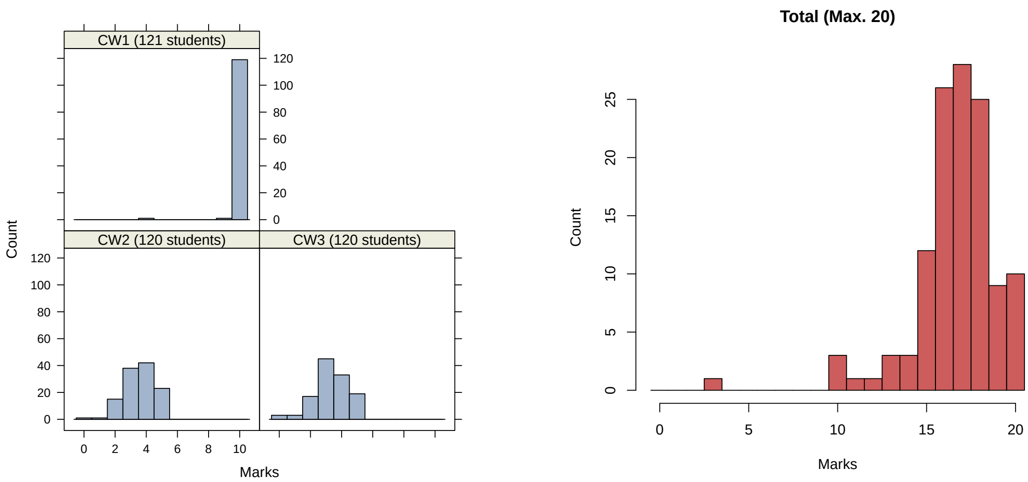

Figure 1: *Histograms for the individual questions and for the total score (right). The histograms above include marks for 121 students for the online component (CW1) and 120 for the written component (CW2 and CW3).*

Overall Remarks

- Generally speaking, people did well: the average for the whole assignment was around 16.6/20 or, approximately 83%.
- I marked the written component papers with an eye to good mathematical style (see the question-by-question notes below for details). To get full marks you needed to make an argument that was technically correct, rigorous, clear and concise.

Remarks about individual problems

(1) (Computing $P(G, k)$: 10 marks)**.**
As I'd hoped, almost everyone got the full 10 marks here, and most students managed to do so in fewer than 5 attempts:

Number of attempts	Number of students making that many attempts
1	41
2	42
3	22
4	11
5	5

Figure 2 shows how the number of attempts made per hour varied across the two-week period during which the online component of the coursework was available. Most of the attempts happened in the evening, except for the last day, when people seem to have worked through the night from Thurs. 4 to Fri. 5 April.

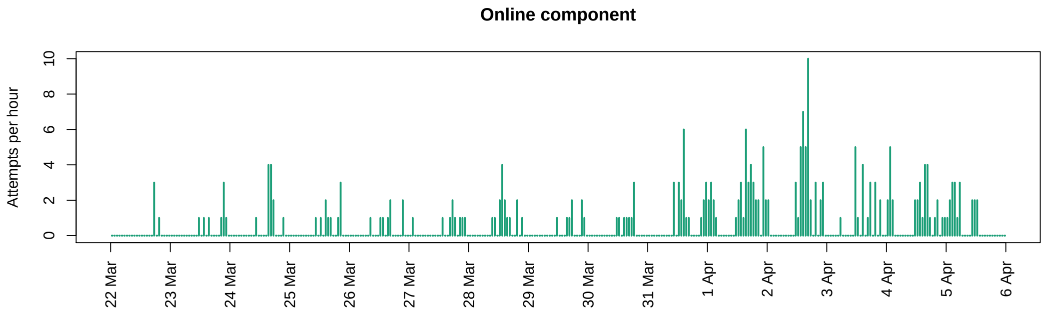

Figure 2: *Number of attempts per hour, over the two-week period when the coursework was available: tick marks appear just after midnight for the named day.*

(2) (Chromatic polynomials of trees: 5 marks).
I hoped that people would have done very well on the online component of the coursework, so I marked the proofs in the written section rather strictly.

- Most successful answers were proofs by induction, typically on the number of vertices $n = |V|$. Oddly, quite a few students felt the need to handle both the $n = 1$ and $n = 2$ cases as Base Cases, even though $n = 1$ is sufficient: the graph with $|V| = 1$ is certainly connected and acyclic, so it is the simplest tree.

- Many students handled the inductive step by removing a leaf vertex, which is a fine strategy: my model answer does the same. But not everyone explained why a tree must *have* at least one leaf (one way to prove this is to invoke a lemma from the lecture on Trees and Forests) and, if this explanation was missing, I deducted a mark.

- Quite a few students offered a semi-algorithmic proof that constructed the colouring one vertex at a time. Most of these proofs were insufficiently detailled to get full marks, lacking one or more of the following ingredients:

  - a concrete procedure to choose the next vertex to be coloured;
  - an argument to explain why every vertex would be coloured (this follows from the connectedness of the tree, but needs to be stated);
  - an argument to explain why the construction can never encounter a vertex with two or more previously-coloured neighbours. This last point follows from the absence of cycles, but needed to be stated explicitly.

Finally, a small minority of students offered incorrect inductive proofs based on the idea of “building up” from a graph covered by the inductive hypothesis to a graph with one more vertex. This issue is discussed in Section 6.2.1 of the lecture notes and, for example, in the Feedback for 2017’s coursework, which says:

> But the problem mentioned in the previous paragraph ...appears to be an example of a common conceptual mistake about how proofs by induction in graphs work: one doesn’t “build up” from graphs that satisfy the inductive hypothesis, but rather starts with a graph that (a) satisfies the hypotheses of the desired theorem and (b) is *one step bigger* than those covered by the inductive hypothesis. One then looks for a way to “cut down” the new graph in some way that makes the inductive hypothesis applicable.

(3) (Multiple connected components: 5 marks).
The result in this question can seem so easy—once you understand it—that it's hard to see what there is to prove. And yet in practice I don't think it is that simple. When I make similar arguments in lecture, for example when counting spregs in the lecture about the proof of Tutte's Matrix-Tree theorem, students don't seem to find them obvious at all: that's why I included the problem in the coursework. There are two key ideas:

- Given a graph $G(V,E)$ with r connected components, $G_1(V_1,E_) \dots G_r(V_r,E_r)$, there is a natural bijection between r-tuples of subgraph colourings
$$(\phi_1, \dots, \phi_r),$$
where $\phi_j : V_j \to \{1,\dots,k\}$ is a k-colouring of G_j.
- The product $\prod_{j=1}^{r} P(G_j,k)$ counts the number of such r-tuples.

A few students offered proofs that contained these ideas (or something equivalent), but most just said imprecise things such as

- "The chromatic polynomial is …quite simply the product of the numbers of colours each vertex can take." This just about makes sense when, as in the proof of Lemma 1 in the coursework, one is constructing the colouring one vertex at a time, but it is not true in general.
- "The connected components are disjoint" They aren't. *Sets* can be disjoint, and the vertex sets V_j *are* pairwise disjoint, but graphs aren't sets and so cannot, without further explanation, be said to be disjoint.
- "The colourings of the components are independent" It's not clear, without further definition, what "independent" means here. The sense one needs is something like "colourings of subgraphs can be combined arbitrarily when constructing colourings of G" and remarks along these lines received more partial credit.

Quite a few students offered an interesting proof by induction whose inductive step involved adding an edge between two components to create a graph with one less connected component. Many of these proofs were good and got substantial partial credit, but almost all of them missed a subtlety that meant that argument failed to work for the step from $r=1$ to $r=2$. This isn't a disaster—one just has to prove a slightly more elaborate base case—but few students who chose this approach seem to have realised that.

Finally, here is one proof of the result that does include the level of detail I was looking for:

Model answer:
One can prove this lemma by induction on r, the number of connected components. Clearly the result is true for $r=1$. Now suppose that it is also true for all values of r up to and including some $r_0 \geq 1$ and consider a graph $G(V,E)$ with r_0+1 components, say, $G_1(V_1,E_1), \dots, G_{r_0+1}(V_{r_0+1},E_{r_0+1})$.

Divide G into two subgraphs: one—call it $G'(V',E')$—that consists of the first r_0 components and another that consists of $G_{r_0+1}(V_{r_0+1},E_{r_0+1})$. Now suppose we have one k-colouring each for these two subgraphs. That is, we have a k-colouring $\phi_1 : V' \mapsto \{1,\dots,k\}$ for G' and a second k-colouring, $\phi_2 : V_{r_0+1} \mapsto \{1,\dots,k\}$, for G_{r_0+1}. Then we can define a k-colouring $\phi : V \mapsto \{1,\dots,k\}$ for the whole graph G by
$$\phi(v) = \begin{cases} \phi_1(v) & \text{if } v \in V' \\ \phi_2(v) & \text{if } v \in V_{r_0+1} \end{cases}$$
And as there are no edges between G' and G_{r_0+1}, we can use this approach to combine arbitrary pairs of subgraph colourings to yield new, distinct colourings of G.

Furthermore, given any k-colouring of G, we can generate a pair of colourings ϕ_1, ϕ_2 like the ones described above by restricting the domain of ϕ to, respectively, V' and V_{r_0+1}. Thus there is a bijection between k-colourings of G and pairs of k-colourings (ϕ_1,ϕ_2) on the two subgraphs.

The number of such pairs (ϕ_1, ϕ_2)—and so also the number of k-colourings of G—is

$$\begin{aligned} P(G, k) &= P(G', k) P(G_{r_0+1}, k) \\ &= \left(\prod_{j=1}^{r_0} P(G_j, k) \right) P(G_{r_0+1}, k) \\ &= \prod_{j=1}^{r_0+1} P(G_j, k) \end{aligned}$$

In passing for the first to the second line, I have used the fact that G' is a graph with r_0 connected components and hence covered by the inductive hypothesis.