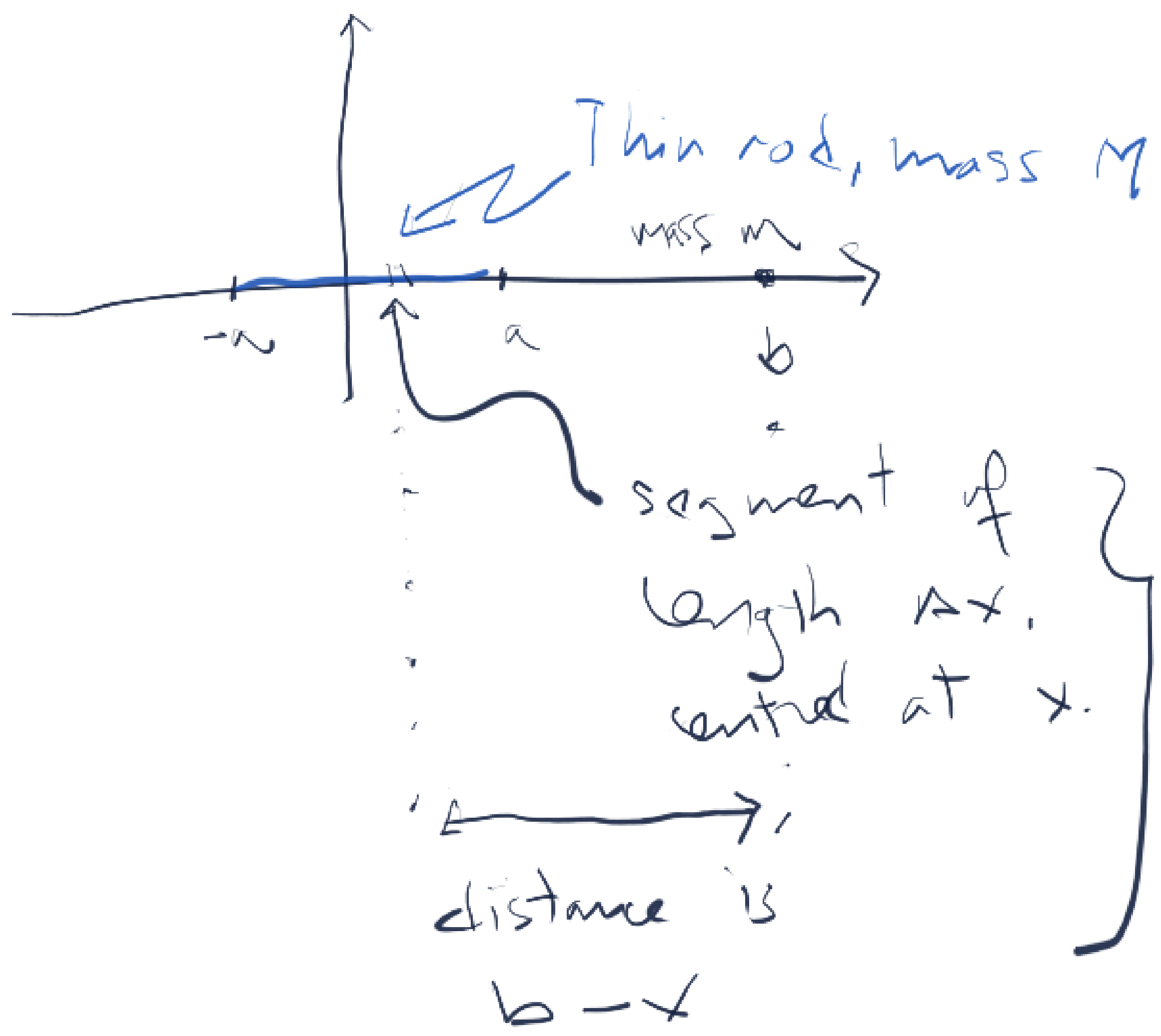
Math 10222: Friday 27 March, 2020



Idea: chop rod into small segments

Tiny segment exerts gravitational force

$$\frac{Gm}{(b-x)^2} \cdot \frac{M\,\Delta x}{2a}$$

← mass of rod (M)

← length of rod ($2a$)

$\frac{\Delta x}{2a}$ = fraction of whole rod in segment.

Contribution to total force due to segment at x $\Bigg\}$ $\frac{Gm}{(b-x)^2} \frac{\Delta x\, M}{2a}$

← Mass of Point at b

← mass of rod

Send $\Delta x \to 0$, integrate to sum contributions

$$\int_{-a}^{a} \frac{GmM}{(b-x)^2} \frac{dx}{2a} = G\frac{mM}{2a}\int_{-a}^{a} \frac{dx}{(b-x)^2}$$

$$= G\frac{mM}{2a}\left[\frac{1}{b-x}\right]_{-a}^{a} = G\frac{mM}{2a}\left[\frac{1}{b-a} - \frac{1}{b+a}\right]$$

$$= G\frac{mM}{2a}\left(\frac{(b+a)-(b-a)}{(b+a)(b-a)}\right) = G\frac{mM}{2a}\left(\frac{2a}{b^2-a^2}\right)$$

$$= G\frac{mM}{b^2-a^2}$$

Consider limit $b \gg a$: P v. far from rod

$$G\frac{mM}{(b^2-a^2)} = G\frac{mM}{b^2}\frac{1}{(1-a^2/b^2)} \approx G\frac{mM}{b^2}\left(1 + a^2/b^2 - \frac{a^4}{b^4} + \dots\right)$$

where I've used

$$\frac{1}{1-\varepsilon} \approx 1 + \varepsilon - \varepsilon^2 \dots$$

Sign of quadratic correction to $F = \frac{GmM}{b^2}$ (the large-b approx) makes sense as closest bit of bar is nearer P than the origin, exerts stronger pull.

Week 4, Problem Q4:

$$\ddot{x} = -g\sin\alpha - \gamma\dot{x} \Rightarrow \ddot{x} + \gamma\dot{x} = -g\sin\alpha$$

Homog:

$$\ddot{x} + \gamma\dot{x} = 0$$

sol'n's are $A + Be^{-\gamma t}$

particular sol'n

$x(t) = ct$?

$$\dot{x} = c,\ \ddot{x} = 0 \Rightarrow \gamma c = -g\sin\alpha \Rightarrow c = \frac{-g}{\gamma}\sin\alpha$$

general sol'n

$$x(t) = A - \frac{g}{\gamma}\sin\alpha\, t + Be^{-\gamma t}$$

Use initial data to find A, B:

$$x(0) = 0 = A + B$$

$$\dot{x}(0) = -B\gamma - \frac{g\sin\alpha}{\gamma} = v$$

$$\Rightarrow \quad B = -\left(v + \frac{g\sin\alpha}{\gamma}\right)/\gamma$$

$$A + B = 0 \Rightarrow A = -B$$

$$x(t) = (1 - e^{-\gamma t})\left(v + \frac{g\sin\alpha}{\gamma}\right)/\gamma - \frac{g\sin\alpha}{\gamma}\, t$$

Check $\ddot{x}$ in, say, m/sec^2 $\Rightarrow$ $\gamma \sim 1/sec$

(so $\ddot{x} + \gamma\dot{x}$ has consistent units)