Overview

Today we'll discuss ways to learn how to think about events that are influenced by chance.

- **Basic probability:** cards, coins and dice
- **Definitions and rules:** *mutually exclusive* events and *independent* events
- **Expectation:** given probabilites, what can we compute?
- **Conditional probability:** for example, the probability that a child smokes, given that her parents do.
- **More applications:** why it's very hard to detect rare things.

What does probability mean?

To say that an event has probability p means that the long-term average of

$$\frac{\text{Number of times event occurs}}{\text{Number of times it could have occured}}$$

is p.

Example 0.1 (A fair coin)

- *Two possible outcomes: Heads and Tails*
- *Each assumed equally likely, so*

$$P(\textit{Heads}) = P(\textit{Tails}) = 1/2$$

Properties of probabilities

- Probabilities are numbers $0 \leq p \leq 1$.
- Given an exhaustive list of possible outcomes, their probabilities add up to one.
- The pair "A happens" and "A doesn't happen" are exhaustive, so

$$P(\text{not } A) = 1 - P(A)$$

Mutually exclusive events

Two events are *mutually exclusive* if one precludes the other, for example: “Toss a coin and get Heads” and “Get Tails on the same toss”.

Example 0.2 (Drawing cards)
Consider drawing a card from an ordinary deck: what it the probability of getting an ace?

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Example 0.2 (Drawing cards)
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Answer

$$\frac{\text{Number of aces}}{\text{Number of cards in deck}} = \frac{4}{52} = \frac{1}{13}.$$

Addition rule for mutually exclusive events

The previous example suggests a rule for working out the probability of either of two mutually exclusive events happening: *If A & B are mutually exclusive events,*

$$P(A \text{ or } B) = P(A) + P(B).$$

Example 0.2 (Rolling a die)
A single roll of a die may show a 1 or a 2, but not both. The probability that it shows either a 1 or *a 2 is*

$$1/6 + 1/6 = 1/3.$$

Independent events

Two events are *independent* if knowing that one has happened tells us nothing about whether the other will happen.

Example 0.2 (Tossing two coins)
Consider tossing a penny and a pound coin.

- Use h & t to show the result for the penny, and H & T, for the pound.
- Possible outcomes are {hH, hT, tH, tT}. Each is equally likely.

Multiplication rule for independent events

- By counting it is clear that
  - P(Heads on penny) = (2/4) = 0.5
  - P(Heads on pound) = (2/4) = 0.5
  - P(Heads on both) = (1/4) = 0.25

Example suggests a rule for probability of two independent events happening together: *If A & B are independent events,*

$$P(A \text{ and } B) = P(A) \times P(B).$$

More about combining events

Finally, there is a rule for combining the probabilities of events that are not mutually exclusive (*i.e.* those for which $P(A$ & $B) \neq 0$).

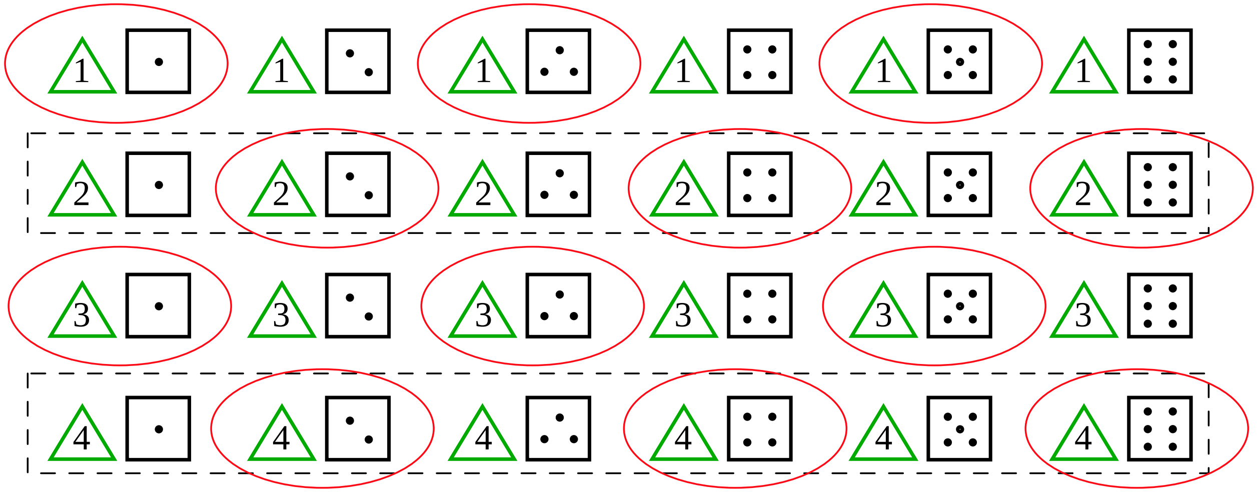
Generally, P(A or B) = P(A) + P(B) - P(A & B).

To see how this works consider rolling two dice, one six-sided and one four-sided and consider events

A The four-sided die comes up an even number;

B The sum of the two rolls is an even number.

Outcomes for the two dice



- Outcomes contributing to event A appear in dashed boxes.
- Those contributing to event B are circled.
- Six outcomes contribute to *both* events.

Using the rule

Counting up events from the diagram, one can see the rule in action

$$
\begin{aligned}
P(A \text{ or } B) &= P(A) + P(B) - P(A \text{ and } B) \\
&= (12/24) + (12/24) - (6/24) \\
&= (1/2) + (1/2) - (1/4) \\
&= (3/4)
\end{aligned}
$$

Review: mutually exclusive events

If A & B are mutually exclusive, which of the following statements are true?

a) $P(A \text{ or } B) = P(A) + P(B)$

b) $P(A \text{ and } B) = 0$

c) $P(A \text{ and } B) = P(A) \times P(B)$

d) $P(A) = P(B)$

e) $P(A) + P(B) = 1$

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Answer: *only a) and b) are true.*

Review: independence

The probability of a certain hard-to-manufacture chip having fault X is 0.20 while the probability of it having flaw Y is 0.05. If these probs are independent, which of the following is true?

a) prob. it has both faults is 0.01;

b) prob. it has both faults is 0.25;

c) prob. it has either fault, or both, is 0.24;

d) if it has X, prob. it has Y also is 0.01;

e) if it has Y, prob. it has X also is 0.20.

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d) if it has X, prob. it has Y also is 0.01;

e) if it has Y, prob. it has X also is 0.20.

Answer: *a), c) and e) are true.*

Conditional probability

Want a concise notion/notation for the probability that one event occurs, given that another has.

Example 0.2 *Roll a six-sided die: what is the probability of getting a two, given that the result is an even number?* There are three possible even numbers, { 2, 4, 6 } and only one of them is a 2, so by direct counting the probability is (1/3).

The notation P(A|B)

Write conditional probabilities as $P(A|B)$ and read them as "the probability of A given B". Examples include:

- $P($ It will rain tomorrow $|$ it is raining now $)$
- $P($ It will rain tomorrow $|$ one is in Manchester $)$
- $P($ Woman gets breast cancer $|$ mother and sister did $)$

A sum rule

The simplest rule about conditional probabilities underlies reasonable statements such as:

$$P(\text{ rain } | \text{ Manchester }) + P(\text{ no rain } | \text{ Manchester }) = 1.$$

More formally, the rule is

> *If one has an exhaustive list of mutually exclusive events then their conditional probabilities add up to one.*

Recovering ordinary probabilities

Sometimes one needs to pass from conditional probabilities back to non-conditional ones. The main tool one needs is the formula:

$$P(A\&B) = P(A|B) \times P(B) \qquad (\star)$$

Using conditional probability

Epidemiology of lung cancer

- Divide subjects into three groups
  - *Heavy smokers:* more that 40 a day
  - *Smokers:* up to 39 per day
  - *Non-smokers:* none
- Find risk of cancer for each group, e.g.

$$P(\text{lung cancer} \mid \text{heavy smoker}).$$

continued . . .

Using . . .

Use conditional probabilities to find risk for general population:

$$\begin{aligned} P(\text{ subject develops lung cancer }) \\ &= P(\text{ [cancer \& heavy smoker] } \textit{or} \\ &\qquad \text{[cancer \& smoker] } \textit{or} \\ &\qquad \text{[cancer \& non-smoker]}) \\ &= P(\text{ cancer \& heavy smoker }) + \\ &\quad P(\text{ cancer \& smoker }) + \\ &\quad P(\text{ cancer \& non-smoker }) \end{aligned}$$

Using . . .

Then use ($\star$) to say

$$\begin{aligned}
&P(\text{ subject develops lung cancer }) = \\
&\qquad P(\text{ cancer } | \text{ heavy smoker }) \times P(\text{ heavy smoker }) + \\
&\qquad P(\text{ cancer } | \text{ smoker }) \times P(\text{ smoker }) + \\
&\qquad P(\text{ cancer } | \text{ non-smoker }) \times P(\text{ non-smoker })
\end{aligned}$$

Bayes Theorem

On the left side of ($\star$) the events A and B play the same role: $A\&B$ means the same thing as $B\&A$. On the right things *seem* to be different: $P(A|B)$ is not generally the same as $P(B|A)$, but

$$\begin{aligned} P(A|B) \times P(B) &= P(A\&B) \\ &= P(B\&A) \\ &= P(B|A) \times P(A) \\ \text{which means} \quad P(A|B) \times P(B) &= P(B|A) \times P(A) \end{aligned}$$

This final expression is sometimes known as *Bayes Theorem*.

Application: screening for rare conditions

Consider a screening program for a CCTV system that observes Manchester's city centre

- target population (say, persons subject to exclusion orders) is rare (1 per 10,000 of population);
- test correctly flags 99% of such persons (true positive);
- test flags only 0.5% of ordinary shoppers (false positive).

What is the probability that when the system identifies a suspect, they really do pose a threat?

Formulate the problem

Using the lanuage of conditional probability

- we want $P($ threat $|$ positive test $)$;
- we have
  - $P($ pos $|$ threat $)$ = 0.99
  - $P($ pos $|$ ordinary $)$ = 0.005
  - $P($ threat $)$ = 0.0001
  - $P($ ordinary $) = (1 - P($ threat $))$ = 0.9999

Start with Bayes Theorem

$$P(\text{ threat } | \text{ pos }) \times P(\text{ pos }) \;=\; P(\text{ pos } | \text{ threat }) \times P(\text{ threat })$$

Solve for necessary probabilities

$$P(\text{ threat } | \text{ pos }) = \frac{P(\text{ pos } | \text{ threat }) \times P(\text{ threat })}{P(\text{ pos })}$$

We need $P(\text{ pos })$, but we can find it with a calculation similar to the one about probability of lung cancer sketched earlier:

$$\begin{aligned} P(\text{ pos }) &= P(\text{ pos } | \text{ threat }) \times P(\text{ threat }) \\ &\quad + P(\text{ pos } | \text{ ordinary }) \times P(\text{ ordinary }) \end{aligned}$$

Assemble results

$$\begin{aligned} & P(\text{ ill} \mid \text{pos }) \\ = & \frac{P(\text{ pos} \mid \text{threat }) \times P(\text{ threat })}{P(\text{ pos })} \\ = & (0.99 \times 0.0001)/(0.99 \times 0.0001 \ + \ 0.005 \times 0.9999) \\ \approx & 0.019 \end{aligned}$$

Discouraging: a positive result from a implausibly precise recognition system gives only a lukewarm indication that a person may pose a problem.