Step 4. Bounds on the Riemann zeta function

We wish to bound $F(s)$. We start from (2), when

$$|F(s)| \leq \left| \frac{\zeta'(s)}{\zeta(s)} \right| + |\zeta(s)|,$$

and give upper bounds on

$$|\zeta(s)|, \quad |\zeta'(s)| \quad \text{and} \quad \left| \frac{1}{\zeta(s)} \right|.$$

We have shown that $\zeta(s)$ has no zeros in $\text{Re} \ s \geq 1$. We will give upper and lower bounds on $\zeta(s)$ and its derivative in the slightly larger region of

$$s = \sigma + it \text{ with } |t| \geq 2 \text{ and } \sigma > 1 - \frac{a}{\log |t|},$$

for any $a > 0$ as long as $\sigma > 1/2$.

Because $\zeta(\sigma - it) = \overline{\zeta(\sigma + it)}$ and thus

$$|\zeta(\sigma - it)| = \left| \overline{\zeta(\sigma + it)} \right| = |\zeta(\sigma + it)|,$$

it suffices to give bounds for $t$ positive. For simplicity write $\eta(t) = a/\log t$.

4.1. Approximate $\zeta(s)$ by a finite sum.

In the next important result we approximate the Riemann zeta function by a finite sum of its Dirichlet series. First recall Theorem 6.11;

$$\sum_{1 \leq n \leq N} \frac{1}{n^s} = 1 + \frac{1}{s-1} + \frac{N^{1-s}}{1-s} - s \int_1^N \{u\} \frac{du}{u^{s+1}},$$

(28)

for $s \neq 1$. Let $N \to \infty$ to get (10):

$$\zeta(s) = 1 + \frac{1}{s-1} - s \int_1^\infty \frac{\{u\}}{u^{1+s}} \, du.$$

We can only take the limit for $\text{Re} \ s > 1$ for then $N^{1-s}/(1-s) \to 0$ as $N \to \infty$. But once the result has been proved we see that the right hand side is defined for $\text{Re} \ s > 0$, $s \neq 1$, becoming the definition of the Riemann zeta function in that larger plane. If we now subtract these last two results we get
Theorem 6.24 For all \( \Re s > 0, s \neq 1 \), and all integers \( N \geq 1 \),

\[
\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} + r_N(s),
\]

(29)

where the remainder is given by

\[
r_N(s) = -s \int_{N}^{\infty} \frac{\{u\}}{u^{s+1}} du
\]

(30)

and satisfies

\[
|r_N(s)| \leq |s| \int_{N}^{\infty} \frac{1}{u^{\sigma+1}} du = \frac{|s|}{\sigma N^{\sigma}}.
\]

Note If you put \( N = 1 \) in Theorem 6.24 you recover Theorem 6.12 (no surprises there) while, if you let \( N \to \infty \), and assume \( \Re s > 1 \) in which case

\[
\lim_{N \to \infty} N^{1-s} = 0,
\]

we recover the Dirichlet Series definition of the zeta function.

The purpose of Theorem 6.24 is to replace the infinite Dirichlet series by a finite series (called a Dirichlet Polynomial) and its strength is the ability to choose an appropriate length of polynomial \( N \), normally depending on \( s \).

4.2 Upper bound on \( \zeta(s) \).

With \( a > 0 \) fixed and \( t > 2 \) we have defined \( \eta(t) = a/\log t \). The important observation to make below is that for \( t \geq 2 \) we have

\[
t^\eta(t) = \exp (\eta(t) \log t) = \exp \left( \frac{a}{\log t} \log t \right) = e^a,
\]

a constant independent of \( t \).

Theorem 6.25 When \( \sigma \geq 1 - \eta(t), \sigma \geq 1/2 \) and \( t \geq 2 \) we have

\[
|\zeta(\sigma+it)| \leq e^a (\log t + 5).
\]

(31)
Proof In Theorem 6.24 with \( t \geq 2 \) given choose \( N = \lfloor t \rfloor \), and estimate each term in (29) separately. The choice of \( N = \lfloor t \rfloor \) with \( t \geq 2 \) implies \( N \geq 2 \) and \( N \leq t < N + 1 \). Then

\[
\left| \sum_{n=1}^{N} \frac{1}{n^s} \right| \leq \sum_{n=1}^{N} \frac{1}{n^n} \leq \sum_{n=1}^{N} \frac{1}{n^{1-\eta(t)}} \leq N^{\eta(t)} \sum_{n=1}^{N} \frac{1}{n} \leq e^a \sum_{n=1}^{N} \frac{1}{n},
\]

since \( \sigma \geq 1 - \eta(t) \) and \( N \leq t \). But, a result often seen in this course, is

\[
\sum_{n=1}^{N} \frac{1}{n} = 1 + \sum_{n=2}^{N} \frac{1}{n} \leq 1 + \int_{1}^{N} \frac{dt}{t} = 1 + \log N.
\]

Also

\[
\left| \frac{N^{1-s}}{s-1} \right| = \frac{N^{1-s}}{|\sigma-1+it|} \leq \frac{N^{\eta(t)}}{|t|} \leq e^a,
\]

since \( t \geq 2 \). Finally

\[
|r_N(s)| \leq \frac{1}{\sigma N^\sigma} \leq \frac{1 + t/\sigma}{N^\sigma} \leq \frac{1 + 2t}{N^{1-\eta(t)}} \quad \text{since } \sigma \geq 1/2
\]

\[
\leq e^a \frac{2N + 3}{N} \quad \text{since } t \leq N + 1
\]

\[
= e^a \left( 2 + \frac{3}{N} \right)
\]

\[
\leq \frac{7}{2} e^a
\]

since \( N \geq 2 \). Combine to get the stated result. \( \blacksquare \)

Note this result, and other bounds on the Riemann zeta function require \( t > 2 \) (and thus \( t < -2 \)). See the appendix for \( |t| \leq 2 \).

4.3 Upper bound on \( \zeta'(s) \)

Next we bound \( |\zeta'(s)| \) from above. You can start by differentiating (28) w.r.t \( s \). Alternatively, if you dislike differentiating under an integral you can
repeat the method in Chapter 1 and apply Partial Summation in

\[
\sum_{1 \leq n \leq N} \frac{\log n}{n^s} = \frac{N \log N}{N^s} - \int_1^N \frac{d}{du} \left( \frac{\log u}{u^s} \right) du + \int_1^N \{u\} \frac{d}{du} \left( \frac{\log u}{u^s} \right) du
\]

\[
= \frac{N^{1-s} \log N}{1-s} - \frac{1}{(1-s)^2} (N^{1-s} - 1) + \int_1^N \{u\} \frac{d}{du} \left( \frac{\log u}{u^s} \right) du,
\]

after integrating by parts a number of times. Thus

\[
- \sum_{n=1}^N \frac{\log n}{n^s} = - \frac{1}{(s-1)^2} - \frac{(1-s) N^{1-s} \log N + N^{1-s}}{(1-s)^2}
\]

\[
- \int_1^N \{u\} \frac{du}{u^{1+s}} + s \int_1^N \{u\} \frac{\log u}{u^{1+s}} du,
\]

for integral \(N \geq 1\) and \(s \neq 1\). Assume \(\text{Re}\ s > 1\) and let \(N \to \infty\) to get.

\[
\zeta'(s) = - \frac{1}{(s-1)^2} - \int_1^\infty \frac{\{u\}}{u^{s+1}} du + s \int_1^\infty \frac{\{u\} \log u}{u^{s+1}} du,
\]

which is what we would have got on differentiating (28) directly. We can see that the integrals here converge for \(\text{Re}\ s > 0\).

Subtracting these last two results gives an approximation to the derivative of the Riemann zeta function by a partial sum of its Dirichlet series,

**Corollary 6.26**

\[
\zeta'(s) = - \sum_{n=1}^N \frac{\log n}{n^s} - \frac{N^{1-s} \log N}{s-1} - \frac{N^{1-s}}{(s-1)^2} - I_1(s) + s I_2(s), \quad (32)
\]

where

\[
I_1(s) = \int_N^\infty \frac{\{u\}}{u^{s+1}} du \quad \text{and} \quad I_2(s) = \int_N^\infty \frac{\{u\} \log u}{u^{s+1}} du.
\]
Leaving it to the student, each term can be estimated, giving

**Theorem 6.27** For $\sigma \geq 1 - \eta(t)$ and $t > 2$ we have

\[
|\zeta'(\sigma + it)| \leq e^a (\log t + 7/4)^2.
\]

**Proof** Exercise.

4.4. Upper bounds for Re $s \geq 1$.

Below we use these upper bounds first for Re $s > 1$. This is equivalent to choosing $a = 0$ in the results above when we then get, for $t \geq 2$,

\[
|\zeta(\sigma + it)| \leq (\log t + 5) \quad \text{and} \quad |\zeta'(\sigma + it)| \leq (\log t + 7/4)^2.
\]

4.5. Lower bound for $\zeta(s)$.

We give an upper bound for $|\zeta^{-1}(\sigma + it)|$ or, equivalently, a lower bound for $|\zeta'(\sigma + it)|$. In fact we go further and bound it both away from 0 and to the left of the line Re $s = 1$. Earlier we proved that $\zeta(s)$ is non-zero in Re $s \geq 1$ but now we will have a region free of zeros to the left of Re $s = 1$, i.e. a zero-free region.

**Lemma 6.28** For $t \geq 2$ and $2 \geq \sigma \geq 1 + \delta(t)$,

\[
|\zeta(\sigma + it)| \geq \frac{1}{2^{15} (\log t + 6)^7},
\]

where

\[
\delta(t) = \frac{1}{2^{19} (\log t + 6)^5}.
\]

**Proof** To get a lower bound in this region start from the important

\[
|\zeta(\sigma)|^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| \geq 1,
\]

valid for $\sigma > 1$. We can apply (34) to the $\zeta(\sigma + 2it)$ term, when

\[
|\zeta(\sigma + 2it)| \leq \log 2t + 5 = \log t + \log 2 + 5 \leq \log t + 6,
\]
say, where 6 is simply chosen as the smallest integer larger than $5 + \log 2$.
For the $\zeta(\sigma)$ term we can recall from Chapter 1 that

$$|\zeta(\sigma)| = 1 + \sum_{n=2}^{\infty} \frac{1}{n^\sigma} \leq 1 + \int_1^{\infty} \frac{dy}{y^\sigma} = 1 + \frac{1}{\sigma - 1} = \frac{\sigma}{\sigma - 1} \leq \frac{2}{\sigma - 1},$$

since $\sigma < 2$. Hence

$$1 \leq |\zeta(\sigma)|^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| \leq \left( \frac{2}{\sigma - 1} \right)^3 |\zeta(\sigma + it)|^4 (\log t + 6),$$

which rearranges as

$$|\zeta(\sigma + it)| \geq \left( \frac{\sigma - 1}{2} \right)^{3/4} \frac{1}{(\log t + 6)^{1/4}} \geq \left( \frac{\delta(t)}{2} \right)^{3/4} \frac{1}{(\log t + 6)^{1/4}}.$$}

The result of the theorem now follows on substituting in $\delta(t)$. ■

The question you should ask, why this choice of $\delta(t)$? Answer, because of the next result. These two results can be combined as one, but since their proofs are so different I have separated them.

**Theorem 6.29** For $t \geq 2$ and $1 - \delta(t) \leq \sigma \leq 1 + \delta(t)$,

$$|\zeta(\sigma + it)| \geq \frac{1}{2^{16} (\log t + 6)^7}.$$}

Note that this is half the size of the lower bound in Lemma 6.28.

**Proof** Write $\sigma_t = 1 + \delta(t)$. We are assuming

$$1 - \delta(t) \leq \sigma < \sigma_t = 1 + \delta(t),$$

and so, for such $\sigma$, we have $0 < \sigma_t - \sigma \leq 2\delta(t)$.

Move along a horizontal line from $\sigma_t + it$ to $\sigma + it$. This time $\sigma$ may be $< 1$ but since $\delta(t) \leq 1/\log t$ we can use the results of Theorem 6.27 with $a = 1$, so $|\zeta'(y + it)| \leq e (\log t + 7/4)^2$ for $y \geq 1 - 1/\log t$.

Then

$$|\zeta(\sigma + it) - \zeta(\sigma_t + it)| = \left| \int_{\sigma_t}^{\sigma} \zeta'(y + it) \, dy \right| \leq e (\sigma_t - \sigma) (\log t + 6)^2.$$

$$\leq 2e\delta(t) (\log t + 6)^2,$$  \hspace{1cm} (37)
by (33), using \( \log t + 7/4 \leq \log t + 6 \) simply so that the bounds in (35) and (37) are comparable.

But how does this upper bound on a difference, (37), give a lower bound on \( \zeta(\sigma + it) \)?

**Idea** If \( w, z \in \mathbb{C} \) and \( |z - w| \) is “small” then \( z \) and \( w \) are ‘about’ the same size. Mathematically, assume \( |z - w| \leq |w| / 2 \). Recall the triangle inequality in the form \( |a - b| \geq |a| - |b| \) for \( a, b \in \mathbb{C} \) (proof \( |a| = |a - b + b| \leq |a - b| + |b| \) by the ‘usual’ form of the triangle inequality. Rearrange to get result.) Using this

\[
|z| = |w - (w - z)| \geq |w| - |w - z| \geq |w| - \frac{|w|}{2} = \frac{|w|}{2},
\]

(38)
i.e. we obtain a lower bound on \( |z| \).

Apply this with \( z = \zeta(\sigma + it) \) and \( w = \zeta(\sigma t + it) \). Then \( |z - w| \leq |w| / 2 \) is satisfied if the upper bound in (37) is less than half the lower bound in (35). That is, if

\[
2e\delta(t)(\log t + 6)^2 \leq \frac{1}{2} \left( \frac{\delta(t)}{2} \right)^{3/4} \frac{1}{(\log t + 6)^{1/4}}.
\]

This rearranges to

\[
\delta(t) \leq \frac{1}{2^{11}e^4 (\log t + 6)^9},
\]

which is satisfied by our choice of \( \delta(t) \) in (36).

From \( |z - w| \leq |w| / 2 \) it follows, by (38), that \( |z| \geq |w| / 2 \), i.e.

\[
|\zeta(\sigma + it)| \geq \frac{1}{2} |\zeta(\sigma t + it)| \geq \frac{1}{2^{16}(\log t + 6)^7}
\]

(39)

by Lemma 6.28.
4.6. Upper bound on $F(s)$.

To combine the three bounds on $\zeta$, $\zeta'$ and $1/\zeta$ they need to be comparable. For this, note that for $\sigma > 1 - 1/\log t$,

$$|\zeta(\sigma+it)| \leq e (\log t + 5) \leq (\log t + 6),$$

$$|\zeta'(\sigma+it)| \leq e \left(\log t + \frac{7}{4}\right)^2 \leq (\log t + 6)^2,$$

are now comparable with the lower bound in Theorem 6.29. Though stated for $t > 2$ they are valid for $|t| > 2$ as long as $t$ is replaced by $|t|$ in the bounds. Hence

**Corollary 6.30** For $2 > \sigma \geq 1 - \delta(t)$ and $|t| > 2$

$$F(\sigma+it) \leq 2^{19} (\log |t| + 6)^9.$$  

**Proof** Looking back at the definition of $F(s)$,

$$|F(\sigma+it)| \leq \frac{|\zeta'(\sigma+it)|}{|\zeta(\sigma+it)|} + |\zeta(\sigma+it)|$$

$$\leq e (\log |t| + 6)^2 2^{16} (\log |t| + 6)^7 + (\log |t| + 6)$$

$$\leq 2^{19} (\log |t| + 6)^9.$$

We in fact only want a weak version of this. For $t > 2$ we have $6 < 8.65\ldots \times \log t$ so $\log t + 6 \leq 9.65\ldots \times \log t$ and thus

$$F(\sigma+it) \ll \log^9 |t|$$

for $t > 2$.

Theorem 6.29 implies that $\zeta(\sigma+it)$ has no zeros in the region

$$\sigma > 1 - \frac{1}{2^{19} (\log t + 6)^9}, |t| \geq 2.$$  

This is called a **zero-free region**. You should draw this region to see how, the larger you take $t$, the less you can go to the left of the $\sigma = 1$ line. No
one has yet proved that there exists $\delta > 0$ such that $\zeta(s)$ has no zeros with $s : \text{Re } s > 1 - \delta$.

The *Riemann Hypothesis* states that $\zeta(s)$ has no zeros with $s : \text{Re } s > 1/2$. It can be shown that this is equivalent to the statement that all zeros $\rho$ of $\zeta(s)$ which satisfy $0 < \text{Re } \rho < 1$ in fact satisfy $\text{Re } s = 1/2$.

Zeros with small imaginary parts.

The above results are valid for $|t| > 2$. What of $|t| \leq 2$?

On $\text{Re } s = 1$ we have $\zeta(s) \neq 0$ and thus there exists $\eta > 0$ such that $|\zeta(s)| > \eta$ when $|t| < 2$. Yet $\zeta(s)$ has a continuation to the half plane $\text{Re } s > 0$, $s \neq 1$ on which it is holomorphic, in particular, continuous. This means there exists $\kappa_1 > 0$ such that $|\zeta(s)| > \eta/2$ when $|t| < 2$ and $1 \geq \sigma > 1 - \kappa_1$. Similarly, it can be shown that $F(s) \ll 1$ when $|t| < 2$ and $1 \geq \sigma > 1 - \kappa_1$, provided $\kappa_1 < 1/2$. (See Additional Notes.)

It is possible, and see Jameson, Proposition 5.3.1, to prove

**Proposition 6.31** $\zeta(s)$ has no zeros in the rectangle

$$\frac{3}{4} \leq \sigma \leq 1 \text{ and } |t| \leq \frac{5}{2}.$$

**Proof** Not given.

Divergence of $\zeta(1+it)$ for $t \neq 0$.

In Chapter 1 it was shown that the series defining $\zeta(s)$ converges absolutely for $\text{Re } s > 1$. In the Problem sheet you are asked to show that the series diverges for $\text{Re } s < 1$. That leaves the question of what happens on the vertical line $\text{Re } s = 1$.

An interesting application of Theorem 6.24 is

**Theorem 6.32**

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+it}}$$

diverges for all $t \in \mathbb{R}$. 

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Proof The result is known if \( t = 0 \). If \( t < 0 \) we can look at the conjugate of the series and assume \( t > 0 \) as we now do.

Rearrange Theorem 6.24 as

\[
\sum_{n=1}^{N} \frac{1}{n^s} = \zeta(s) - \frac{N^{s-1}}{s-1} - r_N(s),
\]

where \( |r_N(s)| \leq |s|/\sigma N^\sigma \). With \( s = 1+it \) and \( t > 0 \), we have

\[
\sum_{n=1}^{N} \frac{1}{n^{1+it}} = \zeta(1+it) + \frac{1}{t} e^{i(\pi/2-t\log N)} + r_N(1+it)
\]

where \( |r_N(1+it)| \leq (1+|t|)/N. \)

As \( N \to \infty \) then \( r_N(1+it) \to 0 \) while the

\[
\zeta(1+it) + \frac{1}{t} e^{i(\pi/2-t\log N)}
\]

are values on the circle, centre \( \zeta(1+it) \), of radius \( 1/t \). This sequence of points do not converge but instead go forever round the circle. Hence the sequence of partial sums \( \sum_{n=1}^{N} n^{-1-it} \) has no limit point as \( N \to \infty \), i.e. the sequence does not converge. This is the definition of the series \( \sum_{n=1}^{\infty} n^{-1-it} \) diverging. 

\[\blacksquare\]