32) From the notes we see that the parts of Theorem 4.2 that concern us are:

Let s and t be two simple non-negative $\mathcal{F}$-measurable functions on $(X, \mathcal{F}, \mu)$ and $E, F \in \mathcal{F}$. Then

(i) $I_E(cs) = cI_E(s)$ *for all* $c \in \mathbb{R}$,
(ii) $I_E(s+t) = I_E(s) + I_E(t)$,
(iii) *If* $s \leq t$ *on* E *then* $I_E(s) \leq I_E(t)$,
(iv) *If* $F \subseteq E$ *then* $I_F(s) \leq I_E(s)$.

Proof As in Lemma 3.7 write

$$s = \sum_{i=1}^{M} a_i \chi_{A_i} = \sum_{i=1}^{M} \sum_{j=1}^{N} a_i \chi_{C_{ij}} \quad \text{and} \quad t = \sum_{j=1}^{N} b_j \chi_{B_j} = \sum_{i=1}^{M} \sum_{j=1}^{N} b_j \chi_{C_{ij}}$$

with $C_{ij} = A_i \cap B_j \in \mathcal{F}$.

(i) Note that $cs = \sum_{i=1}^{M} ca_i \chi_{A_i}$ and so

$$\begin{aligned} I_E(cs) &= \sum_{i=1}^{M} ca_i \mu(A_i) \\ &= c\sum_{i=1}^{M} a_i \mu(A_i) = cI_E(s). \end{aligned}$$

(ii) Note that $s + t = \sum_{i=1}^{M} \sum_{j=1}^{N} (a_i + b_j) \chi_{C_{ij}}$ and so

$$\begin{aligned} I_E(s+t) &= \sum_{i=1}^{M} \sum_{j=1}^{N} (a_i + b_j) \mu(C_{ij} \cap E) \\ &= \sum_{i=1}^{M} \sum_{j=1}^{N} a_i \mu(C_{ij} \cap E) + \sum_{i=1}^{M} \sum_{j=1}^{N} b_j \mu(C_{ij} \cap E) \\ &= \sum_{i=1}^{M} a_i \mu\left(\bigcup_{j=1}^{N} (C_{ij} \cap E)\right) + \sum_{j=1}^{N} b_j \mu\left(\bigcup_{i=1}^{M} (C_{ij} \cap E)\right) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{since } \mu \text{ is additive} \\ &= \sum_{i=1}^{M} a_i \mu(A_i \cap E) + \sum_{j=1}^{N} b_j \mu(B_j \cap E) \\ &= I_E(s) + I_E(t). \end{aligned}$$

(iii) Given any pair $(i,j) : 1 \le i \le M, 1 \le j \le N$ for which $C_{ij} \cap E \neq \phi$, we have for any $x \in C_{ij} \cap E$ that

$$\begin{aligned} a_i &= s(x) \text{ since } x \in C_{ij} \subseteq A_i \\ &\le t(x) \text{ since } s \le t \\ &= b_j \text{ since } x \in C_{ij} \subseteq B_j. \end{aligned}$$

So

$$I_E(s) = \sum_{i=1}^{M} \sum_{j=1}^{N} a_i \mu(C_{ij} \cap E) \le \sum_{i=1}^{M} \sum_{j=1}^{N} b_j \mu(C_{ij} \cap E) = I_E(t).$$

(iv) By the monotonicity of μ we have

$$I_F(s) = \sum_{i=1}^{M} a_i \mu(A_i \cap F) \le \sum_{i=1}^{M} a_i \mu(A_i \cap E) = I_E(s).$$

33) a) Start with

$$n \log \left(1 + \frac{t}{n}\right) = n \int_1^{1+\frac{t}{n}} \frac{dy}{y} = \int_0^t \frac{dx}{1 + \frac{x}{n}} = \int_0^t \frac{ndx}{n+x}.$$

Now $\left\{\frac{n}{n+x}\right\}_{n \ge 1}$ is an increasing sequence of non-negative, Lebesgue integrable functions with limit function $\equiv 1$. So by Lebesgue's Monotonic Convergence Theorem we find that

$$\lim_{n\to\infty} n \log \left(1 + \frac{t}{n}\right) = \lim_{n\to\infty} \int_0^t \frac{ndx}{n+x} = \int_0^t \lim_{n\to\infty} \frac{ndx}{n+x} = t.$$

b) Given $m > n$ then for $x > 0$ we have $mn + mx > mn + nx$ and so

$$\frac{m}{m+x} > \frac{n}{n+x}.$$

Thus

$$m \log \left(1 + \frac{t}{m}\right) = \int_0^t \frac{mdx}{m+x} > \int_0^t \frac{ndx}{n+x} = n \log \left(1 + \frac{t}{n}\right),$$

or

$$\left(1 + \frac{t}{m}\right)^m > \left(1 + \frac{t}{n}\right)^n.$$

Defining

$$g_n(x) = \begin{cases} \left(1+\frac{x}{n}\right)^n e^{-2x} & \text{for } 0 \le x \le n \\ 0 & \text{for } n < x \end{cases}$$

we have an increasing sequence of non-negative Lebesgue measurable functions. The limit function is $g(x) = e^x e^{-2x} = e^{-x}$ for all $x \geq 0$ having used part (a). So by Lebesgue's Monotonic Convergence Theorem we find that

$$\lim_{n\to\infty} \int_0^n \left(1+\frac{x}{n}\right)^n e^{-2x} dx = \int_0^\infty e^{-x} dx = 1.$$

34) a) Starting with $(1-x)^{-2} = 1 + 2x + 3x^2 + ... = \sum_{n=0}^\infty (n+1)\, x^n$, gives

$$\left(\frac{\log x}{1-x}\right)^2 = \sum_{n=0}^\infty (n+1)\, x^n \left(\log x\right)^2.$$

Apply Corollary 4.13 with $f_n(x) = (n+1)\, x^n (\log x)^2$. These are continuous functions on $[0,1]$ and so are Lebesgue measurable. Obviously non-negative so

$$\int_0^1 \left(\frac{\log x}{1-x}\right)^2 dx = \sum_{n=0}^\infty (n+1) \int_0^1 x^n \left(\log x\right)^2 dx.$$

Integrate by parts twice to see that

$$\int_0^1 x^n \left(\log x\right)^2 dx = \frac{2}{(n+1)^3}.$$

Hence

$$\int_0^1 \left(\frac{\log x}{1-x}\right)^2 dx = \sum_{n=0}^\infty \frac{2}{(n+1)^2} = 2\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{3}.$$

b) As in (a) use Corollary 4.13 to justify

$$\int_0^1 \frac{x^p \log x}{1-x} dx = \sum_{n=0}^\infty \int_0^1 x^{p+n} \log x dx.$$

Integrating by parts gives

$$\int_0^1 x^{p+n} \log x dx = \left[\frac{x^{p+n+1}}{p+n+1}\log x\right]_0^1 - \int_0^1 \frac{x^{p+n}}{p+n+1} dx.$$

To deal with the first term at $x = 0$ we require $p+n+1 > 0$ for all $n \geq 0$, that is, $p > -1$. In which case

$$\int_0^1 x^{p+n}\log x dx = 0 - \frac{1}{(p+n+1)}\left[\frac{x^{p+n+1}}{p+n+1}\right]_0^1 = -\frac{1}{(p+n+1)^2}.$$

Hence

$$\int_0^1 \frac{x^p \log x}{1-x}dx = -\sum_{n=0}^{\infty}\frac{1}{(p+n+1)^2} = -\sum_{n=1}^{\infty}\frac{1}{(p+n)^2}$$

as long as $p > -1$.

35) Looking first at those x for which f takes on the values $0, 1$ and 2 we find

$$\begin{aligned} f(x) &= 0 \text{ for either } x \in \mathbb{Q} \text{ or } 0.1 \leq x \leq 1, \\ f(x) &= 1 \text{ for } x \notin \mathbb{Q} \text{ and } 0.01 \leq x < 0.1, \\ f(x) &= 2 \text{ for } x \notin \mathbb{Q} \text{ and } 0.001 \leq x < 0.01, \text{ etc.} \end{aligned}$$

(There might be concern that a number such as $x = 0.01$ can also be written as 0.009999... for which f would give a different value. We need not worry, the collection of such points is such a small set, i.e. one of measure zero, that it would not effect the value of the integral.)

For our sequence of simple functions simply choose

$$f_N(x) = \begin{cases} f(x) & 10^{-N} \leq x \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Since $f(x) \geq 0$ this is an increasing sequence of functions. Each f_N is simple, taking only integral values between 0 and $N-1$. Also the functions are measurable, with

$$\{x : f(x) = 0\} = \mathbb{Q} \cup \left[\tfrac{1}{10}, 1\right)$$

and, for each $1 \leq n \leq N-1$,

$$\{x : f(x) = n\} = \mathbb{Q}^c \cap \left[\tfrac{1}{10^{n+1}}, \tfrac{1}{10^n}\right),$$

which are all measurable sets.

We can quickly evaluate the integral of these simple functions, $I_{[0,1]}(f_n)$, as

$$\begin{aligned}
\sum_{j=1}^{N-1} j\mu\left(\mathbb{Q}\cap\left[\frac{1}{10^{j+1}},\frac{1}{10^j}\right]\right) &= \sum_{j=1}^{N-1} j\left(\frac{1}{10^j}-\frac{1}{10^{j+1}}\right)\\
&= \sum_{j=1}^{N-1}\frac{j}{10^j}-\sum_{j=2}^{N}\frac{(j-1)}{10^j}\\
&= \frac{1}{10}+\sum_{j=2}^{N-1}\frac{j-(j-1)}{10^j}-\frac{N}{10^N}\\
&= \sum_{j=1}^{N-1}\frac{1}{10^j}-\frac{N}{10^N}.
\end{aligned}$$

By Lebesgue's Monotonic Convergence Theorem we get

$$\int_0^1 f d\mu = \lim_{N\to\infty}\left(\sum_{j=1}^{N-1}\frac{1}{10^j}-\frac{N}{10^N}\right)=\sum_{j=1}^{\infty}\frac{1}{10^j}=0.111...=\tfrac{1}{9}.$$

36) Using the definition of ϕ given in the question and the representation of f as a series we see that

$$\begin{aligned}
\phi\left(\bigcup_{n=1}^{\infty}E_n\right) &= \int_{\bigcup_{n=1}^{\infty}E_n} f d\mu = \int_{\bigcup_{n=1}^{\infty}E_n}\left(\sum_{m=1}^{\infty}f_m\right)d\mu\\
&= \sum_{m=1}^{\infty}\int_{\bigcup_{n=1}^{\infty}E_n} f_m d\mu \qquad \text{by Corollary 4.13}\\
&= \sum_{m=1}^{\infty}\int_{E_m} f_m d\mu \qquad \text{since } f_m\equiv 0 \text{ on } E_n \text{ when } n\neq m\\
&= \sum_{m=1}^{\infty}\int_{E_m} f d\mu \qquad \text{since } f_m = f \text{ on } E_m\\
&= \sum_{m=1}^{\infty}\phi(E_m).
\end{aligned}$$

Hence ϕ is σ-additive.

37) Let $x\in\mathbb{R}$ be given. Then

$$\liminf_{n\to\infty} g_n(x) = \lim_{n\to\infty}\left\{\inf_{r\geq n} g_r(x)\right\}.$$

But $g_r(x) \geq 0$ and $g_r(x) = 0$ for all $r \geq x - 1$ (see definition of g_r), so $\inf_{r\geq n} g_r(x) = 0$. True for all x implies $\liminf\limits_{n\to\infty} g_n \equiv 0$ and so $\int_{\mathbb{R}} \liminf\limits_{n\to\infty} g_n d\mu = 0$.

Yet $\int_{\mathbb{R}} g_n d\mu = 1$ for all n and so $\liminf\limits_{n\to\infty} \int_{\mathbb{R}} g_n d\mu = 1$. Thus we have strict inequality.

38) a) Since we have both $0 \leq \max(a,0)$ and $a \leq \max(a,0)$ it follows that

$$\begin{aligned} 0 = 0 + 0 &\leq \max(a,0) + \max(b,0), \\ a + b &\leq \max(a,0) + \max(b,0), \end{aligned}$$

and so

$$\max(a+b, 0) \leq \max(a,0) + \max(b,0).$$

b) Since we have both $\min(a,0) \leq 0$ and $\min(a,0) \leq a$ it follows that

$$\begin{aligned} \min(a,0) + \min(b,0) &\leq 0, \\ \min(a,0) + \min(b,0) &\leq a + b, \end{aligned}$$

and so

$$\min(a,0) + \min(b,0) \leq \min(a+b, 0).$$

c)

$$\begin{aligned} (f+g)^+(x) &= \max((f+g)(x), 0) \\ &= \max(f(x) + g(x), 0) \\ &\leq \max(f(x), 0) + \max(g(x), 0), \qquad \text{by part (a)}, \\ &= f^+(x) + g^+(x). \end{aligned}$$

d)

$$\begin{aligned} (f+g)^-(x) &= -\min((f+g)(x), 0) \\ &\leq -\min(f(x), 0) - \min(g(x), 0), \qquad \text{by part (b)}, \\ &= f^-(x) + g^-(x). \end{aligned}$$

39) If we have $|a - b| = a + b$ where $a, b \geq 0$ we can square both sides to get $|a-b|^2 = (a+b)^2$, in which case, $-2ab = 2ab$. Thus $ab = 0$ and so either

$a = 0$ or $b = 0$. It should be noted that with either of these possibilities we have equality as required.

40) (i) Let

$$f_n(x) = \begin{cases} f(x) & \text{if } x \leq n \\ 0 & \text{if } x > n. \end{cases}$$

Recall from notes that f is Lebesgue integrable if, and only if, $|f\ |$ is Lebesgue integral. Since $|f_n| \leq |f\ |$ this first gives us that $|f_n|$ and thus f_n are Lebesgue integrable. But further, $\lim_{n\to\infty} f_n(x) = f(x)$ with $|f\ |$ is integrable and $|f_n| \leq |f\ |$ means that we can apply Lebesgue's Dominated Convergence Theorem and deduce that

$$\begin{aligned} \int_0^\infty f d\mu &= \lim_{n\to\infty} \int_0^\infty f_n d\mu \\ &= \lim_{n\to\infty} \int_0^n f_n d\mu \text{ since } f_n(x) = 0 \text{ for all } x > n, \\ &= \lim_{n\to\infty} \int_0^n f d\mu. \text{ since } f_n(x) = f(x) \text{ for all } x \leq n. \end{aligned}$$

(ii) We now assume that f is Lebesgue measurable (which is weaker than Lebesgue integrable) and non-negative. The sequence of functions f_n defined in part (i) form an increasing sequence of Lebesgue measurable non-negative functions and so we can apply Lebesgue's Monotonic Convergence Theorem to deduce that

$$\int_0^\infty f d\mu = \lim_{n\to\infty} \int_0^\infty f_n d\mu = \lim_{n\to\infty} \int_0^n f_n d\mu = \lim_{n\to\infty} \int_0^n f d\mu. \qquad (*)$$

41) The function $f(t) = e^{-t}t^{x-1}$ is continuous over $[0, \infty)$ and so is Lebesgue measurable over that interval. We have to show that it is Lebesgue integrable by showing that it's integral over $[0, \infty)$ is finite. We do this by applying (40)(ii) and calculating $\lim_{n\to\infty} \int_0^n f d\mu$. In fact all we do is show that this limit is finite by bounding the integrand $e^{-t}t^{x-1}$ in terms of functions that are easier to integrate. For example, for $t \in [0, 1]$, we have $e^{-t}t^{x-1} \leq e^{-1}t^{x-1}$ and so

$$\int_0^1 e^{-t}t^{x-1} d\mu \leq e^{-1} \int_0^1 t^{x-1} d\mu = e^{-1} \left[\frac{t^x}{x}\right]_0^1 = \frac{e^{-1}}{x} < \infty.$$

For $t \geq 1$ we can find$^{(\dagger)}$ $\kappa = \kappa(x)$ such that $t^{x-1} \leq \kappa e^{t/2}$ in which case $e^{-t}t^{x-1} \leq \kappa e^{-t/2}$ and so

$$\int_1^n e^{-t}t^{x-1}d\mu \leq \kappa \int_1^n e^{-t/2}d\mu < 2\kappa.$$

So the limit in (*) is of an increasing sequence of values of integrals bounded above, hence the limit exists. Thus $e^{-t}t^{x-1}$ is Lebesgue integrable.

$^{(\dagger)}$ Let N be an integer $> x-1$. It suffices to find κ such that $t^N \leq \kappa e^{t/2}$. If $N = 0$ then $\kappa = 1$ will suffice. Assume $N \geq 1$. Expanding, and taking the $j = N$ term,

$$e^{t/2} = \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{t}{2}\right)^j > \frac{1}{N!}\left(\frac{t}{2}\right)^N,$$

since $t \geq 0$. So choose $\kappa = 2^N N!$.

42) The restriction of $t \geq 0$ in question 33(a) was not necessary so that result holds also with $t < 0$, or equivalently,

$$\lim_{n\to\infty} n \log\left(1 - \frac{t}{n}\right) = -t, \text{ i.e. } \lim_{n\to\infty}\left(1 - \frac{t}{n}\right)^n = e^{-t}$$

for $t \geq 0$. Now define

$$f_n(t) = \begin{cases} \left(1 - \frac{t}{n}\right)^n t^{x-1} & \text{if } 0 \leq t < n \\ 0 & \text{otherwise.} \end{cases}$$

As in question 33(b) this is an increasing sequence. First note that

$$n \log\left(1 - \frac{t}{n}\right) = -\int_0^t \frac{ndx}{n-x}.$$

Assume $m > n$ then $mn - mx < mn - nx$ so

$$\frac{m}{m-x} < \frac{n}{n-x}$$

which is well defined since $x \leq t < n$, and so

$$m \log\left(1 - \frac{t}{m}\right) > -\int_0^t \frac{mdx}{m-x} > -\int_0^t \frac{ndx}{n-x} > n \log\left(1 - \frac{t}{n}\right)$$

or

$$\left(1 - \frac{t}{m}\right)^m > \left(1 - \frac{t}{n}\right)^n.$$

Thus we can apply Lebesgue's Monotonic Convergence Theorem to justify the interchange of integration and limit in

$$\begin{aligned}\Gamma(x) &= \int_0^\infty e^{-t}t^{x-1}dt = \int_0^\infty \lim_{n\to\infty} f_n(t)dt = \lim_{n\to\infty}\int_0^\infty f_n(t)dt \\ &= \lim_{n\to\infty}\int_0^n f_n(t)dt \text{ since } f_n(t) = 0 \text{ for all } t > n.\end{aligned}$$

By changes of variables and integrating by parts we see that, as long as $x + j \neq 0$ for all of $0 \leq j \leq n$, we have

$$\begin{aligned}\int_0^n f_n(t)dt &= \int_0^n \left(1 - \frac{t}{n}\right)^n t^{x-1}dt = n^x \int_0^1 (1-y)^n y^{x-1}dy \\ &= n^x \left\{ \left[\frac{(1-y)^n y^x}{x}\right]_0^1 + \frac{n}{x}\int_0^1 (1-y)^{n-1} y^x dy \right\} \\ &= n^x \frac{n}{x}\int_0^1 (1-y)^{n-1} y^x dy = ... \\ &= n^x \frac{n!}{x(x+1)...(x+n-1)}\int_0^1 y^{x+n-1}dy \\ &= \frac{n!n^x}{x(x+1)...(x+n)}.\end{aligned}$$

Hence

$$\Gamma(x) = \lim_{n\to\infty}\frac{n!n^x}{x(x+1)...(x+n)},$$

as long as $x + j \neq 0$ for all $j \in \mathbb{N}\cup\{0\}$, i.e. $-x \notin \mathbb{N}\cup\{0\}$.

43) Starting with $(e^x - 1)^{-1} = e^{-x}(1 - e^{-x})^{-1} = e^{-x}\sum_{j=0}^\infty (e^{-x})^j$ we find that

$$\frac{x^{a-1}}{e^x - 1} = \sum_{n=1}^\infty x^{a-1}e^{-xn},$$

for $x > 0$. This is a sum of non-negative Lebesgue measurable functions. So we can use Corollary 4.13 to justify the interchange in

$$\begin{aligned}
\int_0^\infty \frac{x^{a-1}}{e^x-1}d\mu &= \int_0^\infty \sum_{n=1}^\infty x^{a-1}e^{-xn}d\mu \\
&= \sum_{n=1}^\infty \int_0^\infty x^{a-1}e^{-xn}d\mu \\
&= \sum_{n=1}^\infty \frac{1}{n^a}\int_0^\infty t^{a-1}e^{-t}dt \\
&= \sum_{n=1}^\infty \frac{1}{n^a}\Gamma(a)
\end{aligned}$$

by the definition in Question 42. Note that the equality justified by Corollary 4.13 means that if one side is finite then so are both with the same value and if one side in infinite then so is the other. In our case if $a > 1$ then $\sum_{n=1}^\infty \frac{1}{n^a}$ is convergent and so both sides of our result are finite.

44)(i) Apply Lebesgue's Dominated Convergence Theorem. From Question 33 we have that

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{-n}\sin\left(\frac{x}{n}\right) = e^{-x}\times 0 = 0$$

for all $x > 0$. For a dominating function, h, choose a bound on the $n = 3$ integrand:

$$\left|\left(1+\frac{x}{n}\right)^{-n}\sin\left(\frac{x}{n}\right)\right| \le \left(1+\frac{x}{n}\right)^{-n} \le \left(1+\frac{x}{3}\right)^{-3}$$

for all $n \ge 3$. (Having used the fact proved in question 33 that $\left\{\left(1+\frac{x}{n}\right)^n\right\}_{n\ge 1}$ is an increasing sequence.) The function $h(x) = \left(1+\frac{x}{3}\right)^{-3}$ in integrable over $[0,\infty]$ and so the Dominated Convergence Theorem justifies

$$\begin{aligned}
\lim_{n\to\infty}\int_0^\infty \left(1+\frac{x}{n}\right)^{-n}\sin\left(\frac{x}{n}\right)dx &= \int_0^\infty \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{-n}\sin\left(\frac{x}{n}\right)dx \\
&= \int_0^\infty 0dx = 0.
\end{aligned}$$

(ii) Start from $(1+nx)(1+x) > 1+(n+1)x$ to see that $1+nx > \frac{1+(n+1)x}{1+x}$ and so

$$\frac{1+nx}{(1+x)^n} > \frac{1+(n+1)x}{(1+x)^{n+1}}.$$

Thus the sequence of functions $\left\{\frac{1+nx}{(1+x)^n}\right\}_{n\geq 1}$ is decreasing. Though the terms are non-negative and measurable we cannot use Lebesgue's Monotonic Convergence Theorem directly since it is concerned with increasing sequences. Instead we might hope to use the Dominated Convergence Theorem. But for this we need to know the limit, that the functions are integrable and dominated by an integrable function. The limit is easily seen. If $x = 0$ then all terms in the sequence equal 1 so the limit is 1. If $x > 0$ we start with the observation that the binomial expansion gives $(1+x)^n \geq 1 + nx + \frac{n(n-1)}{2}x^2$ and so

$$\frac{1+nx}{(1+x)^n} \leq \frac{1+nx}{1+nx+\frac{n(n-1)}{2}x^2} \to 0$$

as $n \to \infty$. Thus the limit is 1 if $x = 0$ and 0 elsewhere, that is, 0 a.e.(μ) on $[0, \infty)$. We could choose the dominating function to be the $n = 3$ term, i.e. $h(x) = (1+3x)/(1+x)^3$. It is easily shown that $\int_0^\infty h(x)dx = 2$ and so h is integrable. But also since the sequence of functions is decreasing each function, at least for $n \geq 3$, is integrable. Hence, using the Dominated convergence Theorem we can justify the interchange in

$$\lim_{n\to\infty}\int_0^\infty \frac{1+nx}{(1+x)^n}d\mu = \int_0^\infty \lim_{n\to\infty}\frac{1+nx}{(1+x)^n}d\mu = \int_0^\infty 0d\mu = 0.$$

45)(i) Start from

$$\operatorname{sech} x^2 = \frac{2}{e^{x^2}+e^{-x^2}} = \frac{2}{e^{x^2}\left(1+e^{-2x^2}\right)} = 2\sum_{n=0}^{\infty}(-1)^n e^{-(2n+1)x^2}.$$

We cannot use Corollary 4.13 directly because of the alternating sign. Instead we write the sum as

$$2\sum_{m=0}^{\infty}\left(e^{-(4m+1)x^2} - e^{-(4m+3)x^2}\right),$$

which is now a sum of non-negative measurable functions. So by Corollary 4,13 we now get

$$\begin{aligned}
\int_0^\infty \text{sech } x^2 dx &= 2\sum_{m=0}^\infty \int_0^\infty \left(e^{-(4m+1)x^2} - e^{-(4m+3)x^2}\right) d\mu \\
&= 2\sum_{m=0}^\infty \left(\frac{\sqrt{\pi}}{2\sqrt{4m+1}} - \frac{\sqrt{\pi}}{2\sqrt{4m+3}}\right) \\
&= \sqrt{\pi}\sum_{m=0}^\infty \left(\frac{1}{\sqrt{4m+1}} - \frac{1}{\sqrt{4m+3}}\right) \\
&= \sqrt{\pi}\sum_{n=0}^\infty \frac{(-1)^n}{\sqrt{2n+1}}.
\end{aligned}$$

Here the hint given in the question has been used to get the second line. Also the last line is only a conditionally convergent series (convergent by the alternating sign test) and so the order of summation is important and that is given by the bracketing in the line before.

(ii) As so often seen start with

$$\frac{\cos x}{e^x+1} = \sum_{n=0}^\infty (-1)^n e^{-(n+1)x} \cos x.$$

We hope to use Theorem 4.19 with g_N the N^{th}-partial sum, so

$$\begin{aligned}
|g_N| &= \left|\sum_{n=0}^N (-1)^n e^{-(n+1)x} \cos x\right| \\
&\leq \left|e^{-x}\sum_{n=0}^N \left(-e^{-x}\right)^n\right| = \left|e^{-x}\frac{1-\left(-e^{-x}\right)^{N+1}}{1-e^{-x}}\right| \\
&\leq \frac{2}{e^x+1} \leq \frac{2}{e^x+e^x} \text{ since } e^x \geq 1 \text{ for } x>0, \\
&= e^{-x}.
\end{aligned}$$

So we can apply Lebesgue's Dominated Convergence Theorem with $h(x) = e^{-x}$. Thus

$$\begin{aligned}
\int_0^\infty \frac{\cos x}{e^x+1} dx &= \int_0^\infty \sum_{n=0}^\infty (-1)^n e^{-(n+1)x} \cos x dx \\
&= \sum_{n=0}^\infty (-1)^n \int_0^\infty e^{-(n+1)x} \cos x dx.
\end{aligned}$$

Setting $I_n = \int_0^\infty e^{-nx} \cos x dx$ and integrating by parts twice shows that $I_n = \frac{1}{n} - \frac{1}{n^2} I_n$, that is, $I_n = \frac{n}{n^2+1}$. Thus

$$\int_0^\infty \frac{\cos x}{e^x + 1} dx = \sum_{n=1}^\infty (-1)^{n-1} I_n = \sum_{n=1}^\infty (-1)^{n-1} \frac{n}{n^2+1}.$$

46) Start from

$$\frac{\sinh bx}{\sinh ax} = \frac{e^{bx} - e^{-bx}}{e^{ax} - e^{-ax}} = \frac{e^{bx} - e^{-bx}}{e^{ax}\left(1 - e^{-2ax}\right)} = \left(e^{bx} - e^{-bx}\right) \sum_{n=0}^\infty e^{-(2n+1)ax}.$$

For $x > 0$ this is a sum over non-negative measurable functions and so we can apply corollary 4.13 to deduce

$$\begin{aligned}
\int_0^\infty \frac{\sinh bx}{\sinh ax} dx &= \int_0^\infty \left(e^{bx} - e^{-bx}\right) \sum_{n=0}^\infty e^{-(2n+1)ax} dx \\
&= \sum_{n=0}^\infty \int_0^\infty \left(e^{bx} - e^{-bx}\right) e^{-(2n+1)ax} dx \\
&= \sum_{n=0}^\infty \left[\frac{e^{bx-(2n+1)ax}}{b - (2n+1)a} - \frac{e^{-bx-(2n+1)ax}}{-b - (2n+1)a} \right]_0^\infty \\
&= \sum_{n=0}^\infty \left(\frac{1}{-b - (2n+1)a} - \frac{1}{b - (2n+1)a} \right) \\
&= \sum_{n=0}^\infty \frac{2b}{\left((2n+1)a\right)^2 - b^2}.
\end{aligned}$$

Note that the condition $b < a$ ensures that none of the denominators of terms in the sum are zero.