21) Let $E \in \mathcal{L}$ be given. Assume first that $\mu(E)$ is finite. We have

$$\mu(E) = \mu^*(E) = \inf \sum_{i=1}^{\infty} \mu(A_i)$$

over all covers $E \subseteq \bigcup_{i=1}^{\infty} A_i$, with $A_i \in \mathcal{E}$. Given $\varepsilon > 0$ choose a cover $\{A_i\}_{i \geq 1}$ such that

$$\mu(E) \leq \sum_{i=1}^{\infty} \mu(A_i) < \mu(E) + \frac{\varepsilon}{2}.$$

Yet by Theorem 1.7 we know that each

$$A_i = \bigcup_{j=1}^{n_i} (a_{ij}, b_{ij}]$$

a disjoint union which in turn can be covered by open intervals as in

$$\subseteq \bigcup_{j=1}^{n_i} \left(a_{ij}, b_{ij} + \frac{\varepsilon}{2^{i+1} n_i} \right).$$

Note that we have used the common trick of weighting ε so we get convergent series later. Choose

$$G = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{n_i} \left(a_{ij}, b_{ij} + \frac{\varepsilon}{2^{i+1} n_i} \right) \in \mathcal{U},$$

which is a cover for E. Then

$$\begin{aligned}
\mu(G) &\leq \sum_{i=1}^{\infty} \sum_{j=1}^{n_i} \mu\left(\left(a_{ij}, b_{ij} + \frac{\varepsilon}{2^{i+1} n_i} \right) \right) \quad \text{since } \mu \text{ is subadditive} \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{n_i} \left(\mu((a_{ij}, b_{ij}]) + \frac{\varepsilon}{2^{i+1} n_i} \right)
\end{aligned}$$

since Lebesgue measure of an interval is simply it's length,

$$
\begin{aligned}
&= \sum_{i=1}^{\infty}\sum_{j=1}^{n_i}\mu((a_{ij},b_{ij}]) + \frac{\varepsilon}{2} \\
&= \sum_{i=1}^{\infty}\mu(A_i) + \frac{\varepsilon}{2} \\
&\qquad \text{since the cover of } A_i \text{ is by a } \textbf{disjoint} \text{ union of } (a_{ij},b_{ij}] \\
&< \left(\mu(E) + \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} = \mu(E) + \varepsilon.
\end{aligned}
$$

Hence, since $E \subseteq G$ and the measures are finite we find that $\mu(G \setminus E) = \mu(G) - \mu(E) < \varepsilon$.

Assume now that $\mu(E)$ is infinite.

Decompose $E = \bigcup_{n\in\mathbb{Z}} E_n$ where $E_n = E \cap [n, n+1)$. Then $\mu(E_n)$ is finite and we can follow the argument above but with ε replaced by $\varepsilon/2^{|n|+2}$ to find $G_n \in \mathcal{U}$ with $E_n \subseteq G_n$ and $\mu(G_n \setminus E_n) < \varepsilon/2^{|n|+2}$. Then set $G = \bigcup_{n\in\mathbb{Z}} G_n \in \mathcal{U}$ when

$$
\begin{aligned}
\mu(G \setminus E) &= \mu\left(\bigcup_{n\in\mathbb{Z}}(G_n \setminus E)\right) \le \sum_{n\in\mathbb{Z}}\mu(G_n \setminus E) \\
&\le \sum_{n\in\mathbb{Z}}\mu(G_n \setminus E_n) \le \sum_{n\in\mathbb{Z}}\frac{\varepsilon}{2^{|n|+2}} \\
&= \frac{\varepsilon}{2^2} + 2\sum_{n\ge 1}\frac{\varepsilon}{2^{n+2}} = \frac{\varepsilon}{2^2} + 2\frac{\varepsilon}{2^2} \\
&< \varepsilon.
\end{aligned}
$$

Hence in both cases we can find G.

(*Note in the second part of the proof it was important that we could decompose $\mathbb{R} = \bigcup_{n\in\mathbb{Z}}[n, n+1)$, where each $[n, n+1)$ has finite measure, i.e. $(\mathbb{R}, \mathcal{L}, \mu)$ is σ-finite.)

22) The result is trivial if $c = 0$ so we may assume that $c \neq 0$.

(i) If $I \in \mathcal{P}$ then $I = (a, b]$ for some a and b and

$$
\begin{aligned}
\mu(cI) &= \mu((ca, cb]) \text{ if } c > 0 \\
&= cb - ca \\
&= c(b-a) = c\mu(I).
\end{aligned}
$$

If $c < 0$ then

$$\begin{aligned}\mu(cI) &= \mu([cb, ca)) \\ &= ca - cb \\ &= -c(b-a) = -c\mu(I).\end{aligned}$$

So we see that both cases can be written as $\mu(cI) = |c|\mu(I)$.

(ii) If $E \in \mathcal{E}$ then, by Theorem 1.7, $E = \bigcup_{i=1}^n I_i$, a disjoint union of $I_i \in \mathcal{P}$. By definition of the extended measure given in the proof of Theorem 2.2 we have $\mu(E) = \sum_{i=1}^n \mu(I_i)$ which by part (i) gives $\mu(cE) = |c|\mu(E)$.

(iii) We now look at the outer measure μ^*. Let $A \subseteq \mathbb{R}$. Then there is a map between the covers $\{E_i\}_{i\geq 1} \subseteq \mathcal{E}$ of A and the covers $\{E_i'\}_{i\geq 1} \subseteq \mathcal{E}$ of cA given by $E_i \to cE_i$ and $E_i' \to \frac{1}{c}E_i'$. By (ii) we have that $\mu(E_i) = \frac{1}{|c|}\mu(cE_i)$ and $\mu(E_i') = |c|\mu(\frac{1}{c}E_i')$ and so

$$\begin{aligned}\left\{\sum_i \mu(E_i) : A \subseteq \bigcup_i E_i, E_i \in \mathcal{E}\right\} &= \left\{\frac{1}{|c|}\sum_i \mu(cE_i) : cA \subseteq \bigcup_i cE_i, E_i \in \mathcal{E}\right\} \\ &= \left\{\frac{1}{|c|}\sum_i \mu(E_i') : cA \subseteq \bigcup_i E_i', E_i' \in \mathcal{E}\right\}.\end{aligned}$$

The infimum of the first and third sets are equal, that is,

$$\mu^*(A) = \inf \sum_i \mu(E_i) = \inf \frac{1}{|c|}\sum_i \mu(E_i') = \frac{1}{|c|}\mu^*(cA).$$

Hence $\mu^*(cA) = |c|\mu^*(A)$.

(iv) Let $E \in \mathcal{L}$ and $x \in \mathbb{R}$ be given. Take any test set $A \subseteq \mathbb{R}$. Apply the definition of measurable set to E with test set $\frac{1}{c}A$ to get

$$\begin{aligned}\mu^*\left(\frac{1}{c}A\right) &= \mu^*\left(\left(\frac{1}{c}A\right) \cap E\right) + \mu^*\left(\left(\frac{1}{c}A\right) \cap E^c\right) \\ &= \mu^*\left(\frac{1}{c}(A \cap cE)\right) + \mu^*\left(\frac{1}{c}(A \cap (cE)^c)\right).\end{aligned}$$

By (iii) this gives

$$\frac{1}{|c|}\mu^*(A) = \frac{1}{|c|}\mu^*(A \cap cE) + \frac{1}{|c|}\mu^*(A \cap (cE)^c),$$

in which case

$$\mu^*(A) = \mu^*(A \cap cE) + \mu^*(A \cap (cE)^c)$$

for all test sets $A \subseteq \mathbb{R}$. Hence $cE \in \mathcal{L}$. Of course $\mu^* = \mu$ on $\mathcal{L}$ and so (iii) gives

$$\mu(cE) = \mu^*(cE) = |c|\mu^*(E) = |c|\mu(E)$$

for all $E \in \mathcal{L}$.

23) Simply note that

$$\begin{aligned} x &\in f^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right) \Leftrightarrow f(x) \in \bigcup_{i=1}^{\infty} A_i \\ &\Leftrightarrow f(x) \in A_j \text{ for some } j \geq 1 \\ &\Leftrightarrow x \in f^{-1}(A_j) \text{ for some } j \geq 1 \\ &\Leftrightarrow x \in \bigcup_{i=1}^{\infty} f^{-1}(A_j). \end{aligned}$$

Similarly

$$\begin{aligned} x &\in f^{-1}\left(\bigcap_{i=1}^{\infty} A_i\right) \Leftrightarrow f(x) \in \bigcap_{i=1}^{\infty} A_i \\ &\Leftrightarrow f(x) \in A_i \text{ for all } i \geq 1 \\ &\Leftrightarrow x \in f^{-1}(A_i) \text{ for all } i \geq 1 \\ &\Leftrightarrow x \in \bigcap_{i=1}^{\infty} f^{-1}(A_j). \end{aligned}$$

Finally,

$$\begin{aligned} x &\in f^{-1}(A^c) \Leftrightarrow f(x) \in A^c \\ &\Leftrightarrow f(x) \notin A \\ &\Leftrightarrow x \notin f^{-1}(A) \\ &\Leftrightarrow x \in \left(f^{-1}(A)\right)^c. \end{aligned}$$

24) a) Let $\mathcal{G}$ be the σ-field generated by the intervals $[-\infty, c)$. Then $\mathbb{R}^* \backslash [-\infty, c) = [c, \infty] \in \mathcal{G}$ for all $c \in \mathbb{R}$. Thus $[-\infty, d) \cap [c, \infty] = [c, d) \in \mathcal{G}$ for all $c, d \in \mathbb{R}$.

So $\mathcal{G}$ must contain the smallest σ-field containing $[c, d)$ for all $c, d \in \mathbb{R}$, which from question 14 we know to be $\mathcal{B}$.

Also $\mathcal{G}$ contains $\{+\infty\} = \bigcap_{n\geq 1}[n, +\infty]$ and $\{-\infty\} = \bigcap_{n\geq 1}[-\infty, -n)$.

Hence $\mathcal{G}$ contains the smallest σ-field containing $\mathcal{B}$ and $\{-\infty, +\infty\}$, that is, $\mathcal{B}^*$. So $\mathcal{B}^* \subseteq \mathcal{G}$.

For the reverse set inclusion note that trivially each $[-\infty, c) = \bigcup_{n\geq 1}[-n, c) \cup \{-\infty\} \in \mathcal{B}^*$. So $\mathcal{B}^*$, a σ-field, contains the smallest such σ-field containing these $[-\infty, c)$, i.e. $\mathcal{G} \subseteq \mathcal{B}^*$.

Hence $\mathcal{G} = \mathcal{B}^*$.

b) Just repeat the proof of Theorem 3.3 in the notes. So let $\mathcal{A}$ be the collection of all intervals of the form $[-\infty, c)$. So part (a) implies that $\sigma(\mathcal{A}) = \mathcal{B}^*$.

$$\begin{array}{llll}
f^{-1}(\mathcal{B}^*) \subseteq \mathcal{F} & \text{iff} & f^{-1}(\sigma(\mathcal{A})) \subseteq \mathcal{F} & \\
& \text{iff} & \sigma(f^{-1}(\mathcal{A})) \subseteq \mathcal{F} & \text{by Lemma 3.2,} \\
& \text{iff} & f^{-1}(\mathcal{A}) \subseteq \mathcal{F} & \text{since } \mathcal{F} \text{ is a } \sigma\text{-field, (see question 13)} \\
& \text{iff} & f^{-1}([-\infty, c)) \subseteq \mathcal{F} & \text{for all } c \in \mathbb{R}, \text{ by definition of } \mathcal{A} \\
& \text{iff} & \{x : f(x) < c\} \in \mathcal{F} & \text{for all } c \in \mathbb{R}.
\end{array}$$

25) Recall the definition that $\alpha = \inf x_i$ if, and only if, $\alpha \leq x_i$ for all i and, given $\varepsilon > 0$, there exists j such that $\alpha \leq x_j < \alpha + \varepsilon$. Yet this holds if, and only if, $-\alpha \geq -x_i$ for all i and, given $\varepsilon > 0$ there exists j such that $-\alpha \geq -x_j < -\alpha - \varepsilon$, which is simply the definition that $-\alpha = \sup(-x_i)$.

Hence $-\inf x_i = \sup(-x_i)$. Thus

$$\begin{aligned}
\limsup_{n\to\infty}(-x_n) &= \lim_{n\to\infty}\left\{\sup_{r\geq n}(-x_r)\right\} \\
&= \lim_{n\to\infty}\left\{-\inf_{r\geq n} x_r\right\} \\
&= -\lim_{n\to\infty}\left\{\inf_{r\geq n} x_r\right\}. \\
&= -\liminf_{n\to\infty} x_n.
\end{aligned}$$

26) a) The negation of the given expression is

$$\begin{aligned}
\neg\left(\exists N \geq 1 : \forall n \geq N,\ x \notin A_n\right) &\equiv \forall N \geq 1, \neg\left(\forall n \geq N,\ x \notin A_n\right) \\
&\equiv \forall N \geq 1, \exists n \geq N, \neg\left(x \notin A_n\right) \\
&\equiv \forall N \geq 1, \exists n \geq N,\ x \in A_n \\
&\equiv \forall N \geq 1, x \in \bigcup_{n \geq N} A_n \\
&\equiv x \in \bigcap_{N \geq 1} \bigcup_{n \geq N} A_n \\
&\equiv x \in \limsup_{n\to\infty} A_n.
\end{aligned}$$

So $x \in \limsup_{n\to\infty} A_n$ if, and only if, it is not the case that x is in only finitely many A_n, that is, x is in infinitely many A_n. Thus

$$\limsup_{n\to\infty} A_n = \{x : x \in A_n \text{ for infinitely many } n\}.$$

(b) This time we note that $x \in A_n$ for all but finitely many n if, and only if, there exists N such that $x \in A_n$ for all $n \geq N$. That is

$$\begin{aligned}
x &\in A_n \text{ for all but finitely many } n \\
&\equiv \exists N \geq 1 : \forall n \geq N\ x \in A_n \\
&\equiv \exists N \geq 1\ x \in \bigcap_{n \geq N} A_n \\
&\equiv x \in \bigcup_{N \geq 1} \bigcap_{n \geq N} A_n \\
&\equiv x \in \liminf_{n\to\infty} A_n.
\end{aligned}$$

Thus

$$\liminf_{n\to\infty} A_n = \{x : x \in A_n \text{ for all but finitely many } n\}.$$

(c) Assume $A_1 \subseteq A_2 \subseteq A_3 \subseteq A_4 \subseteq \ldots$. Trivially we have

$$\limsup_{n\to\infty} A_n \text{ and } \liminf_{n\to\infty} A_n \subseteq \bigcup_{k=1}^{\infty} A_k.$$

If now $x \in \bigcup_{k=1}^{\infty} A_k$ then $x \in A_\ell$ for some $l \geq 1$. Then $x \in A_k$ for all $k \geq l$, that is, $x \in A_n$ for infinitely many n and so $x \in \limsup_{n\to\infty} A_n$. Also, x could not be an element only of sets A_k with $k < \ell$ in which case $x \in A_n$ for all but finitely many n and so $x \in \liminf_{n\to\infty} A_n$. Thus we have

$$\bigcup_{k=1}^{\infty} A_k \subseteq \limsup_{n\to\infty} A_n \text{ and } \liminf_{n\to\infty} A_n.$$

Hence we get equality.

(d) Assume that $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \supseteq \ldots$. If $x \in \bigcap_{k=1}^{\infty} A_k$ then $x \in A_k$ for all k and so we have both that $x \in A_n$ for infinitely many n and $x \in A_n$ for all but finitely many n. Thus $x \in \limsup_{n\to\infty} A_n$ and $x \in \liminf_{n\to\infty} A_n$. Hence

$$\bigcup_{k=1}^{\infty} A_k \subseteq \limsup_{n\to\infty} A_n \text{ and } \liminf_{n\to\infty} A_n$$

which leads to equality as before.

(e)

$$\begin{aligned}\left(\limsup_{n\to\infty} A_n\right)^c &= \left(\bigcap_{N\geq 1}\bigcup_{n\geq N} A_n\right)^c = \bigcup_{N\geq 1}\left(\bigcup_{n\geq N} A_n\right)^c \\ &= \bigcup_{N\geq 1}\bigcap_{n\geq N} A_n^c = \liminf_{n\to\infty} A_n^c.\end{aligned}$$

27) So $A_n = (-1/n, 1]$ if n odd and $A_n = (-1, 1/n]$ if n even.

If $x \in A_n$ for all but finitely many n then $x \in (-1/n, 1]$ for all sufficiently large odd n, in which case we must have $x \in [0, 1]$. Also $x \in (-1, 1/n]$ for all sufficiently large even n, in which case we must have $x \in (-1, 0]$. Hence the only possibility is $x = 0$. Since $0 \in (-1/n, 1]$ and $(-1, 1/n]$ for all n we have $0 \in \liminf_{n\to\infty} A_n$. Hence $\liminf_{n\to\infty} A_n = \{0\}$.

Consider now $\limsup_{n\to\infty} A_n$. If $x \in A_n$ for infinitely many n then perhaps $x \in (-1/n, 1]$ for an infinite collection of odd n in which case $x \in [0, 1]$. Or perhaps $x \in (-1, 1/n]$ for an infinite collection of even n in which case $x \in (-1, 0]$. Hence all points in $(-1, 1]$ could lie in $\limsup_{n\to\infty} A_n$. Since $A_n \subseteq (-1, 1]$ for all n there is no chance of any more points in $\limsup_{n\to\infty} A_n$. Hence $\limsup_{n\to\infty} A_n = (-1, 1]$.

28) Let $E_1 = 2\mathbb{N}, E_2 = 4\mathbb{N}$, $E_3 = 8\mathbb{N}$, $E_4 = 16\mathbb{N}$, ..., so in general $E_n = 2^n\mathbb{N}$. Then, given any $m \in \mathbb{Z}$, when $2^n > m$ we find that $m \notin 2^n\mathbb{N}$. Hence $\bigcap_{n\geq 1} E_n = \phi$ and so $\mu\left(\bigcap_{n\geq 1} E_n\right) = 0$, Yet $\mu(E_n) = \infty$ for all n and so $\lim_{n\to\infty} \mu(E_n) \neq \mu\left(\bigcap_{n\geq 1} E_n\right)$.

29) The hint given is to use

Lemma 4.1 *If* $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ *are in* $\mathcal{F}$ *and* $A = \bigcup_{n=1}^{\infty} A_n$ *then*

$$\lim_{n\to\infty} \mu(A_n) = \mu(A).$$

The present example on a decreasing sequence can be converted to this lemma concerning increasing sequences by noting that $E_1 \supseteq E_2 \supseteq E_3 \supseteq E_4 \supseteq ...$ means that

$$E_1 \setminus E_2 \subseteq E_1 \setminus E_3 \subseteq E_1 \setminus E_4 \subseteq ... .$$

Hence, by Lemma 4.1,

$$\lim_{n\to\infty} \mu(E_1 \setminus E_n) = \mu\left(\bigcup_{n=1}^{\infty}(E_1 \setminus E_n)\right). \tag{a}$$

Note that since $\mu(E_1) < \infty$ and $E_n \subseteq E_1$ we can say that $\mu(E_1 \setminus E_n) = \mu(E_1) - \mu(E_n)$ (this would not necessarily hold if the sets had infinite measure) while

$$\begin{aligned}\bigcup_{n=1}^{\infty}(E_1 \setminus E_n) &= \bigcup_{n=1}^{\infty}(E_1 \cap E_n^c) = E_1 \cap \bigcup_{n=1}^{\infty} E_n^c \\ &= E_1 \cap \left(\bigcap_{n=1}^{\infty} E_n\right)^c = E_1 \setminus \left(\bigcap_{n=1}^{\infty} E_n\right).\end{aligned}$$

And so

$$\mu\left(\bigcup_{n=1}^{\infty}(E_1 \setminus E_n)\right) = \mu\left(E_1 \setminus \left(\bigcap_{n=1}^{\infty} E_n\right)\right) = \mu(E_1) - \mu\left(\bigcap_{n=1}^{\infty} E_n\right).$$

Substituting into (a) gives

$$\lim_{n\to\infty}(\mu(E_1) - \mu(E_n)) = \mu(E_1) - \mu\left(\bigcap_{n=1}^{\infty} E_n\right)$$

which gives the stated result.

30) (a) Throughout this question $F(a-)$ will denote the limit from the left, i.e. $\lim_{\delta\to 0,\delta>0} F(a-\delta)$.

$$
\begin{aligned}
\mu_F([a,b]) &= \mu_F\left(\bigcap_{n=1}^{\infty}\left(a-\tfrac{1}{n},b\right]\right) \\
&= \lim_{n\to\infty}\mu_F\left(\left(a-\tfrac{1}{n},b\right]\right) \qquad \text{by question 29,} \\
&= \lim_{n\to\infty}\left(F(b)-F\left(a-\tfrac{1}{n}\right)\right) \\
&= F(b)-F(a-).
\end{aligned}
$$

$$
\begin{aligned}
\mu_F([a,b)) &= \mu_F\left(\bigcup_{n=1}^{\infty}\left[a,b-\tfrac{1}{n}\right]\right) \\
&= \lim_{n\to\infty}\mu_F\left(\left[a,b-\tfrac{1}{n}\right]\right) \qquad \text{by Lemma 4.1,} \\
&= \lim_{n\to\infty}\left(F(\left(b-\tfrac{1}{n}\right)-F\left(a-\right)\right) \qquad \text{by previous part} \\
&= F(b-)-F(a-).
\end{aligned}
$$

$$
\begin{aligned}
\mu_F((a,b)) &= \mu_F\left(\bigcup_{n=1}^{\infty}\left(a,b-\tfrac{1}{n}\right]\right) \\
&= \lim_{n\to\infty}\mu_F\left(\left(a,b-\tfrac{1}{n}\right]\right) \qquad \text{by Lemma 4.1,} \\
&= \lim_{n\to\infty}\left(F\left(b-\tfrac{1}{n}\right)-F\left(a\right)\right) \\
&= F(b-)-F(a).
\end{aligned}
$$

b) Try

$$
F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 \le x < 1 \\ 2 & \text{if } 1 \le x. \end{cases}
$$

Then, using the results from part (a),

$$
\begin{aligned}
\mu_F((0,1)) &= F(1-)-F(0) = 1-1 = 0, \\
F(1)-F(0) &= 2-1 = 1 \\
\mu_F([0,1]) &= F(1)-F(0-) = 2-0 = 2.
\end{aligned}
$$

Hence

$$\mu_F((0,1)) < F(1) - F(0) < \mu_F([0,1])$$

as required.
31)(a)

$$\mu_F(\{2\}) = \mu_F([2,2]) = F(2) - F(2-) = 9 - 6 = 3.$$

(b)

$$\mu_F([-1/2,3)) = F(3-) - F(-1/2-) = 9 - \left(1 - \tfrac{1}{2}\right) = 7\tfrac{1}{2}.$$

(c)

$$\begin{aligned} \mu_F((-1,0] \cup (1,2)) &= \mu_F((-1,0]) + \mu_F((1,2)) \qquad \text{since } \mu_F \text{ is additive} \\ &= (F(0) - F(-1)) + (F(2-) - F(1)) \\ &= (2-0) + (6-3) \\ &= 5. \end{aligned}$$

d)

$$\begin{aligned} \mu_F([0,1/2) \cup (1,2]) &= \mu_F([0,1/2)) + \mu_F((1,2]) \\ &= (F(1/2-) - F(0-)) + (F(2) - F(1)) \\ &= \left(2\tfrac{1}{4} - 1\right) + (9-3) \\ &= 7\tfrac{1}{4}. \end{aligned}$$

e)

$$\begin{aligned} \left\{x : |x| + 2x^2 > 1\right\} &= \{x : x > 1/2\} \cup \{x : x < -1/2\} \\ &= \bigcup_{n \geq 1} \left(\tfrac{1}{2}, n\right) \cup \bigcup_{m \geq 1} \left(-m, \tfrac{1}{2}\right). \end{aligned}$$

So question 29 implies that

$$\mu_F(\left\{x : |x| + 2x^2 > 1\right\}) = \lim_{n \to \infty} \mu_F\left(\left(\tfrac{1}{2}, n\right)\right) + \lim_{m \to \infty} \mu_F\left(\left(-m, \tfrac{1}{2}\right)\right).$$

But

$$\mu_F\left(\left(\tfrac{1}{2}, n\right)\right) = F(n-) - F(1/2) = 9 - 2\tfrac{1}{4} = 6\tfrac{3}{4}, \text{ for all } n \geq 3.$$

And

$$\mu_F\left(\left(-m, \tfrac{1}{2}\right)\right) = F(-1/2-) - F(-m) = \tfrac{1}{2} - 0 = \tfrac{1}{2}, \text{ for all } m \geq 1.$$

Hence

$$\mu_F(\{x : |x| + 2x^2 > 1\}) = 7\tfrac{1}{4}.$$