4.4 Integration of measurable functions.

Let $(X, \mathcal{F}, \mu)$ be a measure space. If f is $\mathcal{F}$-measurable then we can write $f = f^+ - f^-$ where $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$ are non-negative $\mathcal{F}$-measurable functions. So by definition both $\int_E f^+ d\mu$ and $\int_E f^- d\mu$ exist for all $E \in \mathcal{F}$.

Definition If at least one of these integrals is finite, define the *integral of f on E relative to μ* to be

$$\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu.$$

If $\int_E f d\mu$ is finite we say that f is *μ-integrable* on E. The set of all functions integrable on E will be denoted by $\mathcal{L}_E(\mu)$.

Note With the notation above, if $|f| = |f|^+ - |f|^-$ then $|f|^- \equiv 0$ and $|f|^+ = f^+ + f^-$.

From the definition f is integrable if, and only if, $\int_E f^+ d\mu$ and $\int_E f^- d\mu$ are finite, that is, if $|f| = f^+ + f^-$ is integrable. So if $f \in \mathcal{L}_E(\mu)$ then $|f| \in \mathcal{L}_E(\mu)$. This is more restrictive than for Riemann integration.

Theorem 4.16 *Let $f, g \in \mathcal{L}(\mu)$ and $A \in \mathcal{F}$. Then*

*(i) $f \in \mathcal{L}_A(\mu)$,

*(ii) $af \in \mathcal{L}(\mu)$ *and* $\int_X af d\mu = a \int_X f d\mu$ *for all* $a \in \mathbb{R}$,

(iii) $f + g \in \mathcal{L}(\mu)$ *and* $\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu$,

(iv) *If* $f = 0$ *a.e.*(μ) *then* $\int_X f d\mu = 0$,

(v) *If* $f \leq g$ *a.e.*(μ) *then* $\int_X f d\mu \leq \int_X g d\mu$,

(vi) *If* $f = g$ *a.e.*(μ) *then* $\int_X f d\mu = \int_X g d\mu$.

Proof

*(i) $f \in \mathcal{L}(\mu)$ implies that $\int_X f^+ d\mu$ and $\int_X f^- d\mu$ are finite. But f^+ and f^- are non-negative so we can apply Theorem 4.4(ii) to conclude that $\int_A f^\pm d\mu \leq \int_X f^\pm d\mu < \infty$. Hence $f \in \mathcal{L}_A(\mu)$.

*(ii) Suppose that $a \geq 0$. Then $(af)^\pm = af^\pm$ and so

$$\begin{aligned} \int_X (af)^\pm d\mu &= \int_X af^\pm d\mu = a \int_X f^\pm d\mu \qquad \text{by Theorem 4.4(i)}, \\ &< \infty. \end{aligned}$$

Since $f \in \mathcal{L}(\mu)$ both of $\int_X f^\pm d\mu$ are finite, hence both of $\int_X (af)^\pm d\mu$ are finite, that is, $af \in \mathcal{L}(\mu)$. Further

$$
\begin{aligned}
\int_X af\,d\mu &= \int_X (af)^+ d\mu - \int_X (af)^- d\mu \\
&= a\left(\int_X f^+ d\mu - \int_X f^- d\mu\right) \\
&= a\int_X f\,d\mu. \qquad (1)
\end{aligned}
$$

Suppose $a = -1$ then $(-f)^{\pm} = f^{\mp}$ so $-f$ is integrable and

$$
\begin{aligned}
\int_X (-f)\,d\mu &= \int_X (-f)^+ d\mu - \int_X (-f)^- d\mu \\
&= \int_X f^- d\mu - \int_X f^+ d\mu \\
&= -\int_X f\,d\mu. \qquad (2)
\end{aligned}
$$

Suppose that $a < 0$ then $af = -|a|f$ and so

$$
\begin{aligned}
\int_X af\,d\mu &= \int_X -|a|f\,d\mu = -\int_X |a|f\,d\mu \quad \text{by (2)} \\
&= -|a|\int_X f\,d\mu \quad \text{by (1)} \\
&= a\int_X f\,d\mu.
\end{aligned}
$$

(iii) Starting from the trivial observations that $a \le \max(a, 0)$ and $0 \le \max(a, 0)$ it is easy to show that $\max(a + b, 0) \le \max(a, 0) + \max(b, 0)$ for any reals a and b. Thus $(f + g)^{\pm} \le f^{\pm} + g^{\pm}$ and so

$$
\int_X (f+g)^{\pm} d\mu \le \int_X \left(f^{\pm} + g^{\pm}\right) d\mu = \int_X f^{\pm} d\mu + \int_X g^{\pm} d\mu < \infty,
$$

since f and g are μ-integrable. Hence $f + g$ is μ-integrable. We now look at $f + g$ in two ways:

$$
\begin{aligned}
f + g &= (f + g)^+ - (f + g)^- \\
f + g &= (f^+ - f^-) + (g^+ - g^-).
\end{aligned}
$$

On equating and rearranging,

$$(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+.$$

Both side are sums of non-negative $\mathcal{F}$-measurable functions so, by Theorem 4.12, which says that integrals of sums of such functions equal the sums of integrals, we get

$$\begin{aligned} & \int_X (f+g)^+ d\mu + \int_X f^- d\mu + \int_X g^- d\mu \\ = & \int_X (f+g)^- d\mu + \int_X f^+ d\mu + \int_X g^+ d\mu. \end{aligned}$$

On rearrangement this gives the result.

(iv) The assumption $f = 0$ a.e.(μ) means that there exists a set D of measure zero such that for all $x \in X \setminus D$ we have $f(x) = 0$. In particular $f^{\pm}(x) = 0$ for such x and so $f^{\pm} = 0$ a.e.(μ). Then, by Corollary 4.10 we see that $\int_X f^{\pm} d\mu = 0$ and thus $\int_X f d\mu = 0$.

(v) $f \le g$ a.e.(μ) implies $g - f \ge 0$ a.e.(μ) in which case $(g-f)^- = 0$ a.e.(μ). Write $g = f + (g - f)$ when

$$\begin{aligned} \int_X g d\mu &= \int_X f d\mu + \int_X (g-f)^+ d\mu - \int_X (g-f)^- d\mu \qquad \text{by (iii)} \\ &= \int_X f d\mu + \int_X (g-f)^+ d\mu \qquad \text{by (iv)} \\ &\ge \int_X f d\mu \qquad \text{using } (g-f)^+ \ge 0 \text{ and Theorem 4.4(ii)} \end{aligned}$$

(vi) $f = g$ a.e.(μ) implies $g - f = 0$ a.e.(μ) and so $\int_X (g-f) d\mu = 0$ by part (iv). Hence $\int_X g d\mu = \int_X f d\mu$. ■

Theorem 4.17 *If* $g \in \mathcal{L}(\mu)$ *then*

$$\left| \int_X g d\mu \right| \le \int_X |g| \, d\mu$$

with equality if, and only if, either $g \le 0$ *a.e.*(μ) *or* $g \ge 0$ *a.e.*(μ).

Proof We have seen earlier that $|g| \in \mathcal{L}(\mu)$. Also

$$
\begin{aligned}
\left|\int_X g d\mu\right| &= \left|\int_X g^+ d\mu - \int_X g^- d\mu\right| \\
&\leq \int_X g^+ d\mu + \int_X g^- d\mu \qquad \text{(triangle inequality)} \\
&= \int_X \left(g^+ + g^-\right) d\mu \qquad \left(\text{since } g^+ \text{ and } g^- \text{ are non-negative}\right) \\
&= \int_X |g|\, d\mu.
\end{aligned}
$$

We have equality in $|a-b| \leq a+b$, $a, b \geq 0$ if, and only if, either $a = 0$ or $b = 0$. In the present case this means either

$$
\int_X g^+ d\mu = 0 \qquad \text{or} \qquad \int_X g^- d\mu = 0.
$$

From Lemma 4.7 this means that either

$$
\mu\{x : g^+(x) > 0\} = 0 \qquad \text{or} \qquad \mu\{x : g^-(x) > 0\} = 0,
$$

that is, if either $g \leq 0$ a.e.(μ) or $g \geq 0$ a.e.(μ). ■

Recall that for every measurable function, g, the integral is defined if, and only if, both $\int_X g^\pm d\mu$ are finite. A useful way to check this is given in the following.

Corollary 4.18

If g is $\mathcal{F}$-measurable and if there exists $h \in \mathcal{L}(\mu)$ with $|g| \leq h$ a.e.(μ) then $g \in \mathcal{L}(\mu)$.

Proof Since $g^\pm \leq |g|$ we have

$$
\int_X g^\pm d\mu \leq \int_X |g| d\mu \leq \int_X h d\mu < +\infty.
$$

Hence $g \in \mathcal{L}(\mu)$. ■

The next result extends Theorem 4.11 in that we replace the condition that the sequence be non-negative and increasing by one that the sequence be "dominated". The result is equally as important as theorem 4.11.

Theorem 4.19 Lebesgue's Dominated Convergence.

If $\{g_n\}_{n\geq 1}$ is a sequence of $\mathcal{F}$-measurable functions such that $\lim_{n\to\infty} g_n = g$ a.e.(μ) and if $|g_n| \leq h$ for all $n \geq 1$, where $h \in \mathcal{L}(\mu)$ then

$$
\lim_{n\to\infty} \int_X g_n d\mu = \int_X g d\mu.
$$

Proof Corollary 4.18 implies that $g_n \in \mathcal{L}(\mu)$ for all n. But also $|g_n| \le h$ implies that $|g| \le h$ a.e.(μ) and so, again by Corollary 4.18, $g \in \mathcal{L}(\mu)$.

Consider the sequence $\{h + g_n\}_{n\ge 1}$ of non-negative integrable functions. Fatou's Lemma implies

$$\int_X (h+g)d\mu \le \liminf_{n\to\infty} \int_X (h+g_n)d\mu$$

and so

$$\int_X g d\mu \le \liminf_{n\to\infty} \int_X g_n d\mu.$$

Consider next the sequence $\{h - g_n\}_{n\ge 1}$ of non-negative integrable functions. Fatou's Lemma implies

$$\int_X (h-g)d\mu \le \liminf_{n\to\infty} \int_X (h-g_n)d\mu$$

and so

$$-\int_X g d\mu \le \liminf_{n\to\infty} \int_X (-g_n)d\mu$$

or

$$\int_X g d\mu \ge \limsup_{n\to\infty} \int_X g_n d\mu.$$

Then

$$\int_X g d\mu \le \liminf_{n\to\infty} \int_X g_n d\mu \le \limsup_{n\to\infty} \int_X g_n d\mu \le \int_X g d\mu$$

and so we have equality throughout. In particular $\lim_{n\to\infty} \int_X g_n d\mu$ exists and equals $\int_X g d\mu$. ■

Example 19 Evaluate

$$\lim_{n\to\infty} \int_\alpha^\infty \frac{n^2 x e^{-n^2x^2}}{1+x^2} dx$$

when

(a) $\alpha > 0$,
(b) $\alpha = 0$.

Solution Let $y = nx$ when the integral becomes

$$\int_{n\alpha}^{\infty} \frac{ye^{-y^2}}{1+y^2/n^2} dy = \int_0^{\infty} \chi_{[n\alpha,\infty)}(y) \frac{ye^{-y^2}}{1+y^2/n^2} dy$$

where $\chi_{[n\alpha,\infty)}(y) = 1$ if $y \in [n\alpha, \infty)$, zero otherwise. For each n the integrand is less than or equal to ye^{-y^2} which is Lebesgue integrable over $[0, \infty)$. Hence we can use Theorem 4.19 to interchange the limit and the integration. So

$$\lim_{n\to\infty} \int_{\alpha}^{\infty} \frac{n^2xe^{-n^2x^2}}{1+x^2} dx = \int_0^{\infty} \lim_{n\to\infty} \chi_{[n\alpha,\infty)}(y) \frac{ye^{-y^2}}{1+y^2/n^2} dy.$$

(a) If $\alpha > 0$ then $\chi_{[n\alpha,\infty)}(y) = 0$ as soon as $n > y/\alpha$. So the (pointwise) limit of the integrand is zero and, thus, the integral is zero.

(b) If $\alpha = 0$ then $\chi_{[n\alpha,\infty)}(y) = 1$ for all y and all n. So the limit of the integrand is ye^{-y^2} and thus the value of the integral is 1/2. ■

Theorem 4.19 leads to important results on the interchanging of sums and integrals in the manner of Corollary 4.13.

Theorem 4.20 *Let $\{f_n\}$ be a sequence of integrable functions satisfying*

$$\sum_{n=1}^{\infty} \int_X |f_n| d\mu < \infty.$$

Then $\sum_{n=1}^{\infty} f_n$ converges a.e.(μ), its sum is integrable and

$$\sum_{n=1}^{\infty} \int_X f_n d\mu = \int_X \sum_{n=1}^{\infty} f_n d\mu.$$

Proof

We can apply Corollary 4.14 to the sequence of functions $|f_n|$, obtaining

$$\begin{aligned} \int_X \sum_{n=1}^{\infty} |f_n| d\mu &= \sum_{n=1}^{\infty} \int_X |f_n| d\mu \\ &< \infty \quad \text{by assumption.} \end{aligned}$$

So, by Lemma 4.5, we find that $\sum_{n=1}^{\infty} |f_n| < \infty$ a.e.(μ). In particular $\sum_{n=1}^{\infty} f_n$ converges a.e.(μ). For those x at which it converges we have

$$\left| \sum_{n=1}^{\infty} f_n(x) \right| \le \sum_{n=1}^{\infty} |f_n(x)| \quad \text{while} \quad \sum_{n=1}^{\infty} |f_n| \in \mathcal{L}(\mu).$$

Then by Corollary 4.18 we deduce that $\sum_{n=1}^{\infty} f_n \in \mathcal{L}(\mu)$, i.e. it is integrable.

Finally, in the notation of Theorem 4.19, we set $g_k := \sum_{n=1}^{k} f_n$ and $h := \sum_{n=1}^{\infty} |f_n|$, when Lebesgue's Dominated convergence Theorem implies

$$\begin{aligned}
\sum_{n=1}^{\infty} \int_X f_n d\mu &= \lim_{k\to\infty} \sum_{n=1}^{k} \int_X f_n d\mu \\
&= \lim_{k\to\infty} \int_X \sum_{n=1}^{k} f_n d\mu \qquad \text{by Theorem 4.16(iii)} \\
&= \int_X \lim_{k\to\infty} \sum_{n=1}^{k} f_n d\mu \qquad \text{by Theorem 4.19} \\
&= \int_X \sum_{n=1}^{\infty} f_n d\mu.
\end{aligned}$$

■

Example 20 Suppose $f(x)$ is finite and integrable over (a, b) and let $0 < r < 1$ be a fixed number. Then

$$\int_a^b f(x) \frac{\sin x}{1 - 2r\cos x + r^2} dx = \sum_{n=1}^{\infty} r^{n-1} \int_a^b f(x) \sin nx dx.$$

Solution

Let $z = r(\cos x + i \sin x)$ then

$$\begin{aligned}
\frac{1 - \overline{z}}{1 - (z + \overline{z}) + r^2} &= \frac{1 - \overline{z}}{1 - (z + \overline{z}) + z\overline{z}} = \frac{1 - \overline{z}}{(1 - z)(1 - \overline{z})} \\
&= \frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n \\
&= \sum_{n=0}^{\infty} r^n (\cos nx + i \sin nx).
\end{aligned}$$

Equating imaginary parts gives

$$\frac{\sin x}{1 - 2r\cos x + r^2} = \sum_{n=1}^{\infty} r^{n-1} \sin nx.$$

(Don't forget that $\sin 0 = 0$.) So

$$\int_a^b f(x) \frac{\sin x}{1 - 2r\cos x + r^2} dx = \int_a^b f(x) \sum_{n=1}^{\infty} r^{n-1} \sin nx dx.$$

Yet $f \in \mathcal{L}_{[a,b]}(\mu)$ and so $|f| \in \mathcal{L}_{[a,b]}(\mu)$. Also $|f(x)r^{n-1}\sin nx| \leq |f(x)|$ and so by Corollary 4.18 we can deduce that $f(x)r^{n-1}\sin nx \in \mathcal{L}_{[a,b]}(\mu)$. Also

$$\begin{aligned}\sum_{n=1}^{\infty}\int_a^b |f(x)r^{n-1}\sin nx|dx &\leq \sum_{n=1}^{\infty} r^{n-1}\int_a^b |f(x)|dx \\ &= \frac{r}{r-1}\int_a^b |f(x)|dx \\ &< \infty.\end{aligned}$$

So, by Theorem 4.20, we can interchange the sum and integral as required.■

(The following has not been covered in lectures in 2001/2001.)

Definition We say that $\sum_{n=1}^{\infty} f_n$ *converges boundedly* a.e.(μ) on X if there exists $K > 0$ such that

$$\left|\sum_{n=1}^{N} f_n(x)\right| < K$$

for all $x \in X$ and all $N \geq 1$, and $\sum_{n=1}^{\infty} f_n(x)$ exists for almost all $x \in X$.

The following result can be proved.

Theorem 4.21 Let g be an μ-integrable function and $\{f_n\}_{n\geq 1}$ a sequence of μ-integrable functions for which their sum $\sum_{n=1}^{\infty} f_n$ converge boundedly. Then $g\sum_{n=1}^{\infty} f_n$ is μ-integrable and

$$\sum_{n=1}^{\infty}\int_X g f_n d\mu = \int_X g\sum_{n=1}^{\infty} f_n d\mu.$$

Proof Set $g_N = g\sum_{n=1}^{N} f_n$. So by assumption $|g_N| \leq K|g|$ a.e.(μ) for all n and so the result follows from Theorem 4.19. ■