4.3 Interchanging Integrals with other operations.

We now have one of the major results of this course where we interchange an integral with the operation of a limit.

Theorem 4.11 Lebesgue's Monotone Convergence Theorem

Let $E \in \mathcal{F}$ and let $0 \leq f_1 \leq ... \leq f_n \leq f_{n+1} \leq ...$ be an increasing sequence of non-negative $\mathcal{F}$-measurable functions. defined on E. Then

$$\lim_{n\to\infty} \int_E f_n d\mu = \int_E \lim_{n\to\infty} f_n d\mu.$$

Proof

Since for each $x \in E$, $\{f_n(x)\}_n$ is an increasing sequence the limit, possibly ∞, exists. For $x \in E$ define $f(x) = \lim_{n\to\infty} f_n(x)$.

From Theorem 3.6(iv) we see that f is $\mathcal{F}$-measurable on E and so $\int_E f d\mu$ though it might well be $+\infty$. Also $f \geq f_n$ for all n and so

$$\int_E f d\mu \geq \int_E f_n d\mu$$

by Theorem 4.4(ii). But $\left\{\int_E f_n d\mu\right\}_{n\geq 1}$ is also an increasing sequence and so its limit exists and satisfies

$$\int_E f d\mu \geq \lim_{n\to\infty} \int_E f_n d\mu. \tag{16}$$

(*Of course the limit might well be $+\infty$, when necessarily $\int_E f d\mu = +\infty$ and we have equality in (16). Thus we could henceforward assume that the limit is finite though this will not actually be necessary in the following argument.)

For the inequality in the other direction we need a "trick". Take any non-negative simple $\mathcal{F}$-measurable function $0 \leq s \leq f$ and let $0 \leq c < 1$ be given.

Let $E_n = \{x \in E : f_n(x) > cs(x)\} \in \mathcal{F}$ when $E_1 \subseteq E_2 \subseteq ...$. If $x \in E$ then $f(x) \geq s(x) > cs(x)$. Because of the **strict** inequality we can find $m \geq 1$ such that $f_m(x) > cs(x)$ which means that $x \in E_m$. Thus $E \subseteq \bigcup_{n\geq 1} E_n$. Yet $E_n \subseteq E$ for all n and so $\bigcup_{n\geq 1} E_n = E$. Then

$$\begin{aligned}\int_E f_n d\mu &\geq \int_{E_n} f_n d\mu \\ &> \int_{E_n} cs d\mu \\ &= cI_{E_n}(s)\end{aligned}$$

and so $\lim_{n\to\infty}\int_E f_n d\mu \geq cI_E(s)$ by Theorem 4.2(iv). True for all $c < 1$ means that $\lim_{n\to\infty}\int_E f_n d\mu \geq I_E(s)$. Thus $\lim_{n\to\infty}\int_E f_n d\mu$ is **an** upper bound for $\mathcal{I}(f, E)$ for which $\int_E f d\mu$ is the **least** of all upper bounds. Hence

$$\lim_{n\to\infty}\int_E f_n d\mu \geq \int_E f d\mu. \tag{17}$$

Combining (16) and (17) gives our result. ■

*We can see here why the "trick" of introducing c in the proof is necessary.

Consider the case $f \equiv 1$ on E and $f_n \equiv 1 - \frac{1}{n}$ on E. Then $\lim_{n\to\infty} f_n = f$ on E. We now take any simple function $0 \leq s \leq f$. It is allowable to take $s \equiv 1$. Then without c we have $E_n = \{x \in E : s(x) \leq f(x)\} = \phi$ for all $n \geq 1$ and so $\bigcup_{n=1}^{\infty} E_n = \phi$. But we would like $\bigcup_{n=1}^{\infty} E_n = E$. We see that the problem is allowing $s = f$ on a subset of E of non-zero measure, which in turn is possible if, and only if, f is constant on a set of non-zero measure.

***Note** (i) Given a non-negative $\mathcal{F}$-measurable function f then, by Theorem 3.8 there exists a sequence of non-negative simple $\mathcal{F}$-measurable functions s_n increasing to f. Then Theorem 4.11 implies $\int_E f d\mu = \lim_{n\to\infty} I_E(s_n)$. It can be shown that this limit is independent of the sequence of non-negative simple $\mathcal{F}$-measurable functions. This is often taken as a definition of $\int_E f d\mu$.

*(ii) In Riemann integration we approximate f by splitting the **domain** and obtaining lower and upper step functions. In Lebesgue integration it is the **range** that is split obtaining simple functions. This has the big advantage that the domain need not be $\mathbb{R}$, i.e. we can define integration on general measure spaces.

Example 16 Enumerate the rationals in $[0, 1]$ as $r_1, r_2, r_3, \ldots$. Let

$$g_n(x) = \begin{cases} 1 & \text{if } x = r_i \text{ for some } 1 \leq i \leq n \\ 0 & \text{otherwise.} \end{cases}$$

The g_n satisfy Theorem 4.11 but further, they are Riemann integrable with R-$\int_0^1 g_n dx = 0$ for all $n \geq 1$. Yet

$$\lim_{n\to\infty} g_n(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

which is not Riemann integrable. So restricting only to Riemann integration requires extra conditions to ensure that $\lim_{n\to\infty} g_n$ is Riemann integrable (such as uniform convergence).

(*The virtue of Lebesgue integration is that the limit of a sequence of converging, Lebesgue-measurable functions is measurable (Theorem 3.6(iv)) and thus it can be integrated, though the resulting value might be ∞).

Theorem 4.12 *Let $f, g : X \to \mathbb{R}^+$ be $\mathcal{F}$-measurable functions and $E \in \mathcal{F}$. Then*

$$\int_E (f+g)d\mu = \int_E f d\mu + \int_E g d\mu.$$

Proof

By Theorem 3.10 find sequences $\{s_n\}_{n\geq 1}$ and $\{t_n\}_{n\geq 1}$ of non-negative simple $\mathcal{F}$-measurable functions converging to f and g respectively. Then $\{s_n + t_n\}_{n\geq 1}$ is a sequence of non-negative simple $\mathcal{F}$-measurable functions converging to $f+g$ and so

$$\begin{aligned}
\int_E (f+g)d\mu &= \int_E \lim_{n\to\infty} (s_n + t_n)d\mu \\
&= \lim_{n\to\infty} \int_E (s_n + t_n)d\mu \qquad \text{by Theorem 4.11} \\
& \lim_{n\to\infty} I_E(s_n + t_n) \qquad \text{by Proposition 4.3} \\
&= \lim_{n\to\infty} (I_E(s_n) + I_E(t_n)) \qquad \text{by Theorem 4.2(ii)} \\
&= \lim_{n\to\infty} I_E(s_n) + \lim_{n\to\infty} I_E(t_n) \\
&= \lim_{n\to\infty} \int_E s_n d\mu + \lim_{n\to\infty} \int_E t_n d\mu \qquad \text{by Proposition 4.3} \\
&= \int_E \lim_{n\to\infty} s_n d\mu + \int_E \lim_{n\to\infty} t_n d\mu \qquad \text{by Theorem 4.11} \\
&= \int_E f d\mu + \int_E g d\mu.
\end{aligned}$$

■

Our next result concerns interchanging integration with the operation of infinite summation.

Corollary 4.13 *Let $\{f_n\}_{n\geq 1}$ be a sequence of non-negative $\mathcal{F}$-measurable functions defined on $E \in \mathcal{F}$. Then*

$$\int_E \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int_E f_n d\mu.$$

Proof Let $H_k = \sum_{n=1}^{k} f_n$. Then, by induction based on Theorem 4.12,

$$\int_E H_k d\mu = \sum_{n=1}^{k} \int_E f_n d\mu. \tag{18}$$

for all $k \geq 1$. Since $f_n \geq 0$ for all n we see that H_k is an increasing sequence converging to $\sum_{n=1}^{\infty} f_n$. So

$$\begin{aligned}
\int_E \sum_{n=1}^{\infty} f_n d\mu &= \int_E \lim_{k\to\infty} H_k d\mu \\
&= \lim_{k\to\infty} \int_E H_k d\mu \qquad \text{by Theorem 4.11} \\
&= \lim_{k\to\infty} \sum_{n=1}^{k} \int_E f_n d\mu \qquad \text{by (18)} \\
&= \sum_{n=1}^{\infty} \int_E f_n d\mu.
\end{aligned}$$

■

*__Note__ Corollary 4.13 does not require the integral on the left to be finite or the sum on the right to converge, though if the sum is divergent then, because all terms are non-negative, we can say it converges to $+\infty$ by our extended definition of convergence. Then the result says that if either side is infinite then so are both sides.

In the following examples we will use, without proof, the fact that a function which is Riemann integrable over a finite interval is also Lebesgue integrable over the interval with the same limit. (See Appendix 8a.)

Example 17 Show that

$$\int_0^1 \frac{x^{1/3}}{1-x} \log(1/x) dx = 9 \sum_{n=0}^{\infty} \frac{1}{(3n+1)^2}.$$

Solution The integrand is a continuous function on $(0,1)$ and so measurable and Lebesgue integrable when we can look upon the integral as a Lebesgue integral. We might think that the integrand is problematic at $x = 0$ and $x = 1$ but we need only remember that we can change the values of a function on a set of measure without changing the value of the integral. So consider this problem as one of integrating the function

$$\begin{cases} x^{1/3} (1-x)^{-1} \log(1/x) & \text{for } 0 < x < 1, \\ 0 & \text{for } x = 0 \text{ and } x = 1. \end{cases}$$

Write

$$\frac{x^{1/3}}{1-x}\log(1/x) = x^{1/3}\log(1/x)\sum_{n=0}^{\infty} x^n,$$

valid for $0 < x < 1$. Apply the Corollary to get

$$\begin{aligned}\int_0^1 \frac{x^{1/3}}{1-x}\log(1/x)dx &= \sum_{n=0}^{\infty}\int_0^1 x^{n+1/3}\log(1/x)dx \\ &= 9\sum_{n=0}^{\infty}\frac{1}{(3n+1)^2}\end{aligned}$$

on integrating by parts. ∎

Example 18 Let $0 < p, q < \infty$, then

$$\int_0^1 \frac{x^{p-1}}{1+x^q}dx = \left(\frac{1}{p} - \frac{1}{p+q}\right) + \left(\frac{1}{p+2q} - \frac{1}{p+3q}\right) + \dots .$$

Note The sum on the right hand side is not absolutely convergent. For such a conditionally convergent series it is very important in what order the terms are added; it is an interesting result that given a conditionally convergent series and a real number α we can find an order in which to add the terms of the series so that it converges to α. Hence the inclusion of the brackets in this result.

Solution If we expand

$$x^{p-1}(1+x^q)^{-1} = x^{p-1}(1 - x^q + x^{2q} - x^{3q} + x^{4q} - \dots)$$

the terms are, unfortunately, not all non-negative. So it is necessary to group terms, writing this as $\sum_{n=0}^{\infty} f_n(x)$ where

$$f_n(x) = x^{p-1}(x^{2nq} - x^{(2n+1)q}) \geq 0 \text{ on } [0,1].$$

The f_n are continuous and so Lebesgue integrable. Corollary 4.13 gives

$$\begin{aligned}\int_0^1 \frac{x^{p-1}}{1+x^q}dx &= \sum_{n=0}^{\infty}\int_0^1 x^{p-1}(x^{2nq} - x^{(2n+1)q})dx \\ &= \sum_{n=0}^{\infty}\left[\frac{x^{p+2nq}}{p+2nq} - \frac{x^{p+(2n+1)q}}{p+(2n+1)q}\right]_0^1 \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{p+2nq} - \frac{1}{p+(2n+1)q}\right).\end{aligned}$$

$\blacksquare$

Special cases:

The choice $p = 1, q = 1$ gives

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + ... = \int_0^1 \frac{1}{1+x} dx = \log 2,$$

while $p = 1$ and $q = 2$ gives

$$\left(1 - \frac{1}{3}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{9} - \frac{1}{11}\right) + ... = \int_0^1 \frac{1}{1+x^2} dx = \frac{\pi}{4}.$$

***Theorem 4.14** *Let $(X, \mathcal{F}, \mu)$ be a measure space and f a non-negative $\mathcal{F}$-measurable function. Then $\phi(E) = \int_E f d\mu$ is a measure on the measurable space $(X, \mathcal{F})$.*

Further, if also $\int_X f d\mu < \infty$ then for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $A \in \mathcal{F}$ and $\mu(A) < \delta$ then $\phi(A) < \varepsilon$.

(This is a continuity property).

***Proof** (Not needed)

Let $\{E_n\}$ be a collection of disjoint sets from $\mathcal{F}$. Let

$$f_n(x) = \begin{cases} f(x) & \text{if } x \in E_n \\ 0 & \text{if } x \notin E_n, \end{cases}$$

so $f(x) = \sum_{n=1}^{\infty} f_n(x)$ for all $x \in \bigcup_{n=1}^{\infty} E_n$. For each $n \geq 1$ the function f_n is $\mathcal{F}$-measurable. Then

$$\begin{aligned} \phi\left(\bigcup_{n=1}^{\infty} E_n\right) &= \int_{\bigcup_{n=1}^{\infty} E_n} f d\mu = \int_{\bigcup_{n=1}^{\infty} E_n} \sum_{m=1}^{\infty} f_m d\mu \\ &= \sum_{m=1}^{\infty} \int_{\bigcup_{n=1}^{\infty} E_n} f_m d\mu \qquad \text{by Corollary 4.13} \\ &= \sum_{m=1}^{\infty} \int_{E_m} f_m d\mu \\ &= \sum_{m=1}^{\infty} \int_{E_m} f d\mu \qquad \text{since } f_m = f \text{ on } E_m \\ &= \sum_{m=1}^{\infty} \phi(E_m). \end{aligned}$$

Hence ϕ is σ-additive.

Let

$$F_n(x) = \begin{cases} f(x) & \text{if } f(x) \leq n \\ 0 & \text{if } f(x) > n \end{cases}$$

which are $\mathcal{F}$-measurable functions. The F_n are monotonically increasing to f. From Theorem 4.11 we see that

$$\lim_{n\to\infty} \int_X F_n d\mu = \int_X f d\mu.$$

This means that given any $\varepsilon > 0$ there exists N such that

$$0 \leq \int_X f d\mu - \int_X F_N d\mu < \frac{\varepsilon}{2}.$$

Choose $\delta = \varepsilon/2N$. Then if $A \in \mathcal{F}$ satisfies $\mu(A) < \delta$ we have

$$\begin{aligned} \phi(A) &= \int_A f d\mu = \int_A (f - F_N) d\mu + \int_A F_N d\mu \\ &\leq \int_X (f - F_N) d\mu + \int_A N d\mu, \qquad \text{since } F_N \leq N, \\ &< \frac{\varepsilon}{2} + N\mu(A) \\ &< \varepsilon. \end{aligned}$$

■

***Example** If μ is the Lebesgue measure on $\mathbb{R}$ then $e^{-x^2/2}$ is continuous and, therefore, Lebesgue measurable. So Theorem 4.14 implies $\int_A e^{-x^2/2} d\mu$ is a measure on $\mathbb{R}$.

***Definition**

$$\mu^G(A) = \frac{1}{\sqrt{2\pi}} \int_A e^{-x^2/2} d\mu$$

is the *Gaussian measure* on $\mathbb{R}$. Note that $\mu^G(\mathbb{R}) = 1$.

We can extend Theorem 4.11 to sequences that need not be increasing. The next result is important.

Theorem 4.15 Fatou's Lemma. *If $\{g_n\}_{n\geq 1}$ is a sequence of non-negative $\mathcal{F}$-measurable functions and $E \in \mathcal{F}$ then*

$$\int_E \liminf_{n\to\infty} g_n d\mu \leq \liminf_{n\to\infty} \int_E g_n d\mu.$$

Proof

The function $\liminf_{n\to\infty} g_n$ is $\mathcal{F}$-measurable by Theorem 3.6(ii).

Recall $\liminf_{n\to\infty} g_n = \lim_{n\to\infty}(\inf_{r\geq n} g_r)$. Let $h_n = \inf_{r\geq n} g_r$ which we have seen previously is an increasing sequence of functions. So we can apply Lebesgue's |Monotone Convergence Theorem, Theorem 4.11, to deduce

$$\begin{aligned}\lim_{n\to\infty}\int_E h_n d\mu &= \int_E \lim_{n\to\infty} h_n d\mu \\ &= \int_E \liminf_{n\to\infty} g_n d\mu.\end{aligned}$$

Also $h_n = \inf_{r\geq n} g_r \leq g_n$ and so $\int_E h_n d\mu \leq \int_E g_n d\mu$. Therefore

$$\begin{aligned}\lim_{n\to\infty}\int_E h_n d\mu &= \liminf_{n\to\infty}\int_E h_n d\mu \\ &\leq \liminf_{n\to\infty}\int_E g_n d\mu.\end{aligned}$$

Combining we get the required result. ■

This result can be extended slightly by replacing $\liminf_{n\to\infty} g_n = g$ a.e.(μ) and deducing $\int_E g d\mu \leq \liminf_{n\to\infty}\int_E g_n d\mu$.