4 Integration

4.1 Integration of non-negative simple functions

Throughout we are in a measure space \((X, \mathcal{F}, \mu)\).

**Definition** Let \(s\) be a non-negative \(\mathcal{F}\)-measurable simple function so that

\[ s = \sum_{i=1}^{N} a_i \chi_{A_i}, \]

with disjoint \(\mathcal{F}\)-measurable sets \(A_i, \bigcup_{i=1}^{N} A_i = X\) and \(a_i \geq 0\). For any \(E \in \mathcal{F}\) define the integral of \(f\) over \(E\) to be

\[ I_E(s) = \sum_{i=1}^{N} a_i \mu(A_i \cap E), \]

with the convention that if \(a_i = 0\) and \(\mu(A_i \cap E) = +\infty\) then \(0 \times (+\infty) = 0\). (So the area under \(s \equiv 0\) on \(\mathbb{R}\) is zero.)

**Example 13** Consider \(([0, 1], \mathcal{L}, \mu)\). Define

\[ f(x) = \begin{cases} 1 & \text{if } x \text{ rational} \\ 0 & \text{if } x \text{ irrational}. \end{cases} \]

This is a simple function with \(A_1 = \mathbb{Q} \cap [0, 1] \in \mathcal{L}\) and \(A_0\) the set of irrationals in \([0, 1]\) which, as the complement of \(A_1\), is in \(\mathcal{L}\). Thus \(f\) is measurable and

\[ I_{[0,1]}(f) = 1 \mu(\mathbb{Q} \cap [0, 1]) + 0 \mu(\mathbb{Q}^c \cap [0, 1]) = 0, \]

since the Lebesgue measure of a countable set is zero.

**Lemma 4.1**

If \(E_1 \subseteq E_2 \subseteq E_3...\) are in \(\mathcal{F}\) and \(E = \bigcup_{n=1}^{\infty} E_n\) then

\[ \lim_{n \to \infty} \mu(E_n) = \mu(E). \]

(We say that we have an increasing sequence of sets.)

**Proof**

If there exists an \(n\) such that \(\mu(E_n) = +\infty\) then \(E_n \subseteq E\) implies \(\mu(E) = +\infty\) and the result follows.

So assume that \(\mu(E_n) < +\infty\) for all \(n \geq 1\). Then
\[ E = E_1 \cup \bigcup_{n=2}^{\infty} (E_n \setminus E_{n-1}) \]

is a disjoint union. Note that \( E_{n-1} \subseteq E_n \) implies \( E_n = (E_n \setminus E_{n-1}) \cup E_{n-1} \), a disjoint union. So \( \mu(E_n) = \mu(E_n \setminus E_{n-1}) + \mu(E_{n-1}) \). Because the measures are finite we can rearrange as \( \mu(E_n \setminus E_{n-1}) = \mu(E_n) - \mu(E_{n-1}) \). So

\[
\mu(E) = \mu(E_1) + \sum_{n=2}^{\infty} \mu(E_n \setminus E_{n-1})
\]

\[
= \mu(E_1) + \lim_{N \to \infty} \sum_{n=1}^{N} (\mu(E_n) - \mu(E_{n-1}))
\]

(by definition of infinite sum)

\[
= \lim_{N \to \infty} \mu(E_N). 
\]

**Theorem 4.2**

Let \( s \) and \( t \) be two simple non-negative \( \mathcal{F} \)-measurable functions on \((X, \mathcal{F}, \mu)\) and \( E, F \in \mathcal{F} \). Then

(i) \( I_E(cs) = cI_E(s) \) for all \( c \in \mathbb{R} \),

(ii) \( I_E(s + t) = I_E(s) + I_E(t) \),

(iii) If \( s \leq t \) on \( E \) then \( I_E(s) \leq I_E(t) \),

(iv) If \( F \subseteq E \) then \( I_F(s) \leq I_E(s) \),

(v) If \( E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots \) and \( E = \bigcup_{k=1}^{\infty} E_k \) then \( \lim_{k \to \infty} I_{E_k}(s) = I_E(s) \).

**Proof** (Proofs of all parts will be omitted from lectures and left to students. the idea is to write out the simple functions for both \( s \) and \( t \) in terms of common sets \( C_{ij} \) as in the proof of Lemma 3.7.)

As in Lemma 3.7 write

\[ s = \sum_{i=1}^{M} a_i \chi_{A_i} = \sum_{i=1}^{M} \sum_{j=1}^{N} a_i \chi_{C_{ij}} \]

and

\[ t = \sum_{j=1}^{N} b_j \chi_{B_j} = \sum_{i=1}^{M} \sum_{j=1}^{N} b_j \chi_{C_{ij}} \]
with $C_{ij} = A_i \cap B_j \in {\mathcal F}$.

*(i) Note that $cs = \sum_{i=1}^{M} ca_i \chi_{A_i}$ and so

$$I_E(cs) = \sum_{i=1}^{M} ca_i \mu(A_i)$$

$$= c \sum_{i=1}^{M} a_i \mu(A_i) = cI_E(s).$$

*(ii) Then $s + t = \sum_{i=1}^{M} \sum_{j=1}^{N} (a_i + b_j) \chi_{C_{ij}}$. So

$$I_E(s + t) = \sum_{i=1}^{M} \sum_{j=1}^{N} (a_i + b_j) \mu(C_{ij} \cap E)$$

$$= \sum_{i=1}^{M} \sum_{j=1}^{N} a_i \mu(C_{ij} \cap E) + \sum_{i=1}^{M} \sum_{j=1}^{N} b_j \mu(C_{ij} \cap E)$$

$$= \sum_{i=1}^{M} \mu\left(\bigcup_{j=1}^{N}(C_{ij} \cap E)\right) + \sum_{j=1}^{N} \mu\left(\bigcup_{i=1}^{M}(C_{ij} \cap E)\right)$$

$$= \sum_{i=1}^{M} a_i \mu(A_i \cap E) + \sum_{j=1}^{N} b_j \mu(B_j \cap E)$$

$$= I_E(s) + I_E(t).$$

*(iii) Given any $1 \leq i \leq M, 1 \leq j \leq N$ for which $C_{ij} \cap E \neq \phi$ we have for any $x \in C_{ij} \cap E$ that $a_i = s(x) \leq t(x) = b_j$ so

$$I_E(s) = \sum_{i=1}^{M} \sum_{j=1}^{N} a_i \mu(C_{ij} \cap E)$$

$$\leq \sum_{i=1}^{M} \sum_{j=1}^{N} b_j \mu(C_{ij} \cap E)$$

$$= I_E(t).$$

*(iv) By monotonicity of $\mu$ we have
\[ I_F(s) = \sum_{i=1}^{M} a_i \mu(A_i \cap F) \leq \sum_{i=1}^{M} a_i \mu(A_i \cap E) = I_E(s). \]

*(v) From Lemma 4.1 we know that if we have \( E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots \) and \( E = \bigcup_{k=1}^{\infty} E_k \) then \( \lim_{k \to \infty} \mu(E_k) = \mu(E) \). Thus

\[
\lim_{k \to \infty} I_{E_k}(s) = \lim_{k \to \infty} \sum_{i=1}^{M} a_i \mu(A_i \cap E_k) = \sum_{i=1}^{M} a_i \lim_{k \to \infty} \mu(A_i \cap E_k) = \sum_{i=1}^{M} a_i \mu(A_i \cap E) \text{ by Lemma 4.1,}
\]

\[ = I_E(s). \]

4.2 Integration of non-negative measurable functions.

**Definition** If \( f : X \to \mathbb{R}^+ \) is a non-negative \( \mathcal{F} \)-measurable function, \( E \in \mathcal{F} \), then the integral of \( f \) over \( E \) is

\[ \int_{E} f \, d\mu = \sup \{ I_E(s) : s \text{ a simple } \mathcal{F}\text{-measurable function, } 0 \leq s \leq f \}. \]

Of course, if \( E \neq X \) we need only that \( f \) is defined on some domain containing \( E \).

Let \( \mathcal{I}(f, E) \) denote the set

\[ \{ I_E(s) : s \text{ a simple } \mathcal{F}\text{-measurable function, } 0 \leq s \leq f \} \]

so the integral equals \( \sup \mathcal{I}(f, E) \).

**Note** The integral exists for all non-negative \( \mathcal{F} \)-measurable functions though it might be infinite.
If $\int_E f \, d\mu = \infty$ we say the integral is \textit{defined}.

If $\int_E f \, d\mu < \infty$ we say that $f$ is $\mu$--\textit{integrable} or \textit{summable} on $E$.

**Proposition 4.3**

For a non-negative, $\mathcal{F}$-measurable simple function, $t$, we have $\int_E t \, d\mu = I_E(t)$.

**Proof**

Given any simple $\mathcal{F}$-measurable function, $0 \leq s \leq t$ we have $I_E(s) \leq I_E(t)$ by Theorem 4.2(iii). So $I_E(t)$ is an upper bound for $\mathcal{I}(t, E)$ for which $\int_E t \, d\mu$ is the least of all upper bounds. Hence $\int_E t \, d\mu \leq I_E(t)$.

Also, $\int_E t \, d\mu \geq I_E(s)$ for all simple $\mathcal{F}$-measurable function, $0 \leq s \leq t$, and so is greater than $I_E(s)$ for any particular $s$, namely $s = t$. Hence $\int_E t \, d\mu \geq I_E(t)$.

Thus $\int_E t \, d\mu = I_E(t)$. $\blacksquare$

**Example 14** If $f \equiv k$, a constant, then $\int_E f \, d\mu = I_E(f) = k \mu(E)$.

**Theorem 4.4** Throughout, all sets are in $\mathcal{F}$ and all functions are non-negative and $\mathcal{F}$-measurable.

(i) For all $c \geq 0$,

$$\int_E c f \, d\mu = c \int_E f \, d\mu, \quad (15)$$

(ii) If $0 \leq g \leq h$ on $E$ then

$$\int_E g \, d\mu \leq \int_E h \, d\mu,$$

(iii) If $E_1 \subseteq E_2$ and $f \geq 0$ then

$$\int_{E_1} f \, d\mu \leq \int_{E_2} f \, d\mu.$$

**Proof**

(i) If $c = 0$ then the right hand side of (15) is 0 as is the left hand side by Example 14.

Assume $c > 0$.

If $0 \leq s \leq cf$ is a simple $\mathcal{F}$-measurable function then so is $0 \leq \frac{1}{c}s \leq f$. Thus

$$\int_E f \, d\mu \geq I_E\left(\frac{1}{c}s\right) = \frac{1}{c} I_E(s)$$

by Theorem 4.2(i). Hence $c \int_E f \, d\mu$ is an upper bound for $\mathcal{I}(cf, E)$ for which $\int_E cf \, d\mu$ is the least upper bound. Thus $c \int_E f \, d\mu \geq \int_E cf \, d\mu$. 

5
Starting with the observation that if $0 \leq s \leq f$ is a simple $\mathcal{F}$-measurable function then so is $0 \leq cs \leq cf$ we obtain

$$\int_E (cf) \, d\mu \geq I_E (cs) \quad \text{by the definition of } \int_E$$

$$= cI_E (s) \quad \text{by Theorem 4.2(i)}. $$

Hence $\frac{1}{c} \int_E (cf) \, d\mu$ is an upper bound for $\mathcal{I}(f, E)$ for which $\int_E f \, d\mu$ is the least upper bound. Hence $\frac{1}{c} \int_E (cf) \, d\mu \geq \int_E f \, d\mu$, that is, $\int_E cf \, d\mu \geq c \int_E f \, d\mu$.

Combining both inequalities gives our result.

(ii) Let $0 \leq s \leq g$ be a simple, $\mathcal{F}$-measurable function. Then since $g \leq h$ we trivially have $0 \leq s \leq h$ in which case $I_E(s) \leq \int_E hd\mu$ by the definition of integral $\int_E$. Thus $\int_E hd\mu$ is an upper bound for $\mathcal{I}(g, E)$. As in (i) we get $\int_E hd\mu \geq \int_E gd\mu$.

(iii) Let $0 \leq s \leq f$ be a simple, $\mathcal{F}$-measurable function. Then

$$I_{E_1}(s) \leq I_{E_2}(s) \quad \text{by Theorem 4.2(iii)}$$

$$\leq \int_{E_2} f \, d\mu \quad \text{by the definition of } \int_{E_2}. $$

So $\int_{E_2} f \, d\mu$ is an upper bound for $\mathcal{I}(f, E_1)$ and so is greater than the least of all upper bounds. Hence $\int_{E_2} f \, d\mu \geq \int_{E_1} f \, d\mu$. \hfill \blacksquare

**Lemma 4.5**

Assume $E \in \mathcal{F}$, $f \geq 0$ is $\mathcal{F}$-measurable and $\int_E f \, d\mu < \infty$. Set

$$A = \{x \in E : f(x) = +\infty\}.$$

Then $A \in \mathcal{F}$ and $\mu(A) = 0$.

**Proof**

Since $f$ is $\mathcal{F}$-measurable then $f^{-1}(\{\infty\}) \in \mathcal{F}$ and so $A = E \cap f^{-1}(\{\infty\}) \in \mathcal{F}$. Define

$$s_n(x) = \begin{cases} n & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Since $A \in \mathcal{F}$ we deduce that $s_n$ is an $\mathcal{F}$-measurable simple function. Also $s_n \leq f$ and so

$$n\mu(A) = I_E(s_n) \quad \text{by definition of } I_E$$

$$\leq \int_E f \, d\mu \quad \text{by definition of } \int_E$$

$$< \infty \quad \text{by assumption}.$$
True for all $n \geq 1$ means that $\mu(A) = 0$.

Lemma 4.6

If $f$ is $\mathcal{F}$-measurable and non-negative on $E \in \mathcal{F}$ and $\mu(E) = 0$ then $\int_E f d\mu = 0$.

Proof

Let $0 \leq s \leq f$ be a simple, $\mathcal{F}$-measurable function. So $s = \sum_{n=1}^{N} a_n \chi_{A_n}$ for some $a_n \geq 0, A_n \in \mathcal{F}$. Then $I_E(s) = \sum_{n=1}^{N} a_n \mu(A_n \cap E)$. But $\mu$ is monotone which means that $\mu(A_n \cap E) \leq \mu(E) = 0$ for all $n$ and so $I_E(s) = 0$ for all such simple functions. Hence $I(\{f, E\}) = \{0\}$ and so $\int_E f d\mu = \sup I(f, E) = 0$.

Lemma 4.7

If $g \geq 0$ and $\int_E g d\mu = 0$ then

$$\mu\{x \in E : g(x) > 0\} = 0.$$ 

Proof Let $A = \{x \in E : g(x) > 0\}$ and $A_n = \{x \in E : g(x) > \frac{1}{n}\}$. Then the sets $A_n = E \cap \{x : g(x) > \frac{1}{n}\} \in \mathcal{F}$ satisfy $A_1 \subseteq A_2 \subseteq A_3 \subseteq ...$ with $A = \bigcup_{n=1}^{\infty} A_n$. By lemma 4.1 $\mu(A) = \lim_{n \to \infty} \mu(A_n)$. Using

$$s_n(x) = \begin{cases} \frac{1}{n} & \text{if } x \in A_n \\ 0 & \text{otherwise}, \end{cases}$$

so $s_n \leq g$ on $A_n$ we have

$$\frac{1}{n} \mu(A_n) = I_{A_n}(s_n) \leq \int_{A_n} g d\mu \quad \text{by the definition of } \int_{A_n}$$

$$\leq \int_{E} g d\mu \quad \text{Thereom 4.4(iii)}$$

$$= 0 \quad \text{by assumption.}$$

So $\mu(A_n) = 0$ for all $n$ and hence $\mu(A) = 0$.

Definition

If a property $P$ holds on all points in $E \setminus A$ for some set $A$ with $\mu(A) = 0$ we say that $P$ holds almost everywhere ($\mu$) on $E$, written as $a.e. (\mu)$ on $E$.

(*It might be that $P$ holds on some of the points of $A$ or that the set of points on which $P$ does not hold is non-measurable. This is immaterial. But if $\mu$ is a complete measure, such as the Lebesgue-Steltje's measure $\mu_F$, then the situation is simpler. Assume that a property $P$ holds $a.e. (\mu)$ on $E$. The definition says that the set of points, $D$ say, on which $P$ does not hold can be covered by a set of measure zero, i.e. there exists $A : D \subseteq A$ and $\mu(A) = 0$. Yet if $\mu$ is complete then $D$ will be measurable of measure zero.

In this section we are not assuming that $\mu$ is complete.)
So, for example, Lemma 4.7 can be restated as

**Lemma 4.8**

If \( g \geq 0 \) and \( \int_E gd\mu = 0 \) then \( g = 0 \) a.e.(\( \mu \)) on \( E \).

We can extend Theorem 4.4(ii) as follows.

**Theorem 4.9** If \( g, h : X \to \mathbb{R}^+ \) are \( \mathcal{F} \)-measurable functions and \( g \leq h \) a.e.(\( \mu \)) then

\[
\int_E gd\mu \leq \int_E hd\mu.
\]

**Proof**

By assumption there exists a set \( D \subseteq E \), of measure zero, such that for all \( x \in E \setminus D \) we have \( g(x) \leq h(x) \). Let \( 0 \leq s \leq g \) be a simple, \( \mathcal{F} \)-measurable function, written as

\[
s = \sum_{i=1}^{N} a_i \chi_{A_i}, \quad \text{with } \bigcup_{i=1}^{N} A_i = E.
\]

The problem here is that we may well not have \( s \leq h \). Define

\[
s^*(x) = \begin{cases} 
  s(x) & \text{if } x \notin D \\
  0 & \text{if } x \in D 
\end{cases}
\]

\[
= \sum_{i=1}^{N} a_i \chi_{A_i \cap D^c}
\]

which is still a simple, \( \mathcal{F} \)-measurable function. Then for \( x \in E \setminus D \) we have \( s^*(x) = s(x) \leq g(x) \leq h(x) \), while for \( x \in D \) we have \( s^*(x) = 0 \leq h(x) \). Thus \( s^*(x) \leq h(x) \) for all \( x \in E \).

Note that \( A_i = (A_i \cap D^c) \cup (A_i \cap D) \), a disjoint union in which case \( \mu(A_i) = \mu(A_i \cap D^c) + \mu(A_i \cap D) = \mu(A_i) \). But \( A_i \cap D \subseteq D \) and so \( \mu(A_i \cap D) \leq \mu(D) = 0 \). Thus \( \mu(A_i) = \mu(A_i \cap D^c) \). Hence

\[
I_E(s^*) = \sum_{i=1}^{N} a_i \mu(A_i \cap D^c)
\]

\[
= \sum_{i=1}^{N} a_i \mu(A_i)
\]

\[
= I_E(s).
\]
So $I_E(s) = I_E(s^*) \leq \int_E h \, d\mu$ by the definition of integral $\int_E$. Thus $\int_E h \, d\mu$ is an upper bound for $\mathcal{I}(g, E)$ while $\int_E g \, d\mu$ is the least of all upper bounds for $\mathcal{I}(g, E)$. Hence $\int_E h \, d\mu \geq \int_E g \, d\mu$. ■

**Corollary 4.10**

If $g, h : X \to \mathbb{R}^+$ are $\mathcal{F}$-measurable with $g = h$ a.e. (μ) on $E$ then

$$\int_E g \, d\mu = \int_E h \, d\mu.$$  

**Proof**

By assumption there exists a set $D \subseteq E$ of measure zero such that for all $x \in E \setminus D$ we have $g(x) = h(x)$. In particular, for these $x$ we have $g(x) \leq h(x)$ and $h(x) \leq g(x)$. So $g \leq h$ a.e. (μ) on $E$ and $h \leq g$ a.e. (μ) on $E$. Hence the result follows from two applications of Theorem 4.9. ■

So, a function may have its values altered on a set of measure zero without altering the value of its integral. In particular, by Lemma 4.5 we may assume that a non-negative integrable function is finite valued.

**Example 15** (c.f. Example 13) On $([0, 1], \mathcal{L}, \mu)$ the function

$$f(x) = \begin{cases} 
    1 & \text{if } x \text{ is rational} \\
    0 & \text{if } x \text{ irrational}
\end{cases}$$

is 0 a.e. (μ) on $[0, 1]$. So

$$\int_{[0,1]} f \, d\mu = \int_{[0,1]} 0 \, d\mu = 0.$$