3 Measurable Functions

Notation A pair $(X, \mathcal{F})$ where $\mathcal{F}$ is a σ-field of subsets of X is a *measurable space.*

If μ is a measure on $\mathcal{F}$ then $(X, \mathcal{F}, \mu)$ is a *measure space.*

If $\mu(X) < \infty$ then $(X, \mathcal{F}, \mu)$ is a *probability space* and μ a *probability measure.* The measure can, and normally is, renormalised such that $\mu(X) = 1$.

Definition The *extended Borel sets* $\mathcal{B}^*$ of $\mathbb{R}^*$ is the set of unions of sets from $\mathcal{B}$ with subsets of $\{-\infty, +\infty\}$.

I leave it to the student to check that $\mathcal{B}^*$ is a σ-field.

Importantly we have

Proposition 3.1

The σ-field $\mathcal{B}^*$ is the σ-field generated in $\mathbb{R}^*$ by all intervals $(c, +\infty], c \in \mathbb{R}$.

Proof

Let $\mathcal{G}$ be the σ-field generated by these intervals. Then $\mathbb{R}^* \backslash (c, +\infty] = [-\infty, c] \in \mathcal{G}$ for all $c \in \mathbb{R}$. Thus $(c, +\infty] \cap [-\infty, d] = (c, d] \in \mathcal{G}$ for all $c, d \in \mathbb{R}$. So $\mathcal{G}$ must contain the smallest σ-field containing $(c, d]$ for all $c, d \in \mathbb{R}$, namely $\mathcal{B}$.

Also $\mathcal{G}$ contains $\{+\infty\} = \bigcap_{n \geq 1}(n, +\infty]$ and $\{-\infty\} = \bigcap_{n \geq 1}[-\infty, -n]$.

Hence $\mathcal{G}$ contains the smallest σ-field containing $\mathcal{B}$ and $\{-\infty, +\infty\}$, that is, $\mathcal{B}^*$. So $\mathcal{B}^* \subseteq \mathcal{G}$.

Trivially each $(c, +\infty] = \bigcup_{n \geq 1}(c, n] \cup \{+\infty\} \in \mathcal{B}^*$. So the smallest σ-field containing these $(c, +\infty]$ must be contained in any other σ-field containing them, i.e. $\mathcal{G} \subseteq \mathcal{B}^*$.

Hence $\mathcal{G} = \mathcal{B}^*$. $\blacksquare$

Obviously the same σ-field is generated by the intervals $\{[c, +\infty], c \in \mathbb{R}\}$ or $\{[-\infty, c], c \in \mathbb{R}\}$ or $\{[-\infty, c), c \in \mathbb{R}\}$.

*For a justification of the following definitions we might look ahead to how we will integrate functions $f : (X, \mathcal{F}) \to \mathbb{R}^*$. One method is to approximate f by splitting the **range** of f, that is $\mathbb{R}$, into intervals $(a_\nu, a_{\nu+1}]$ and examining the set $\{x \in X : a_\nu < f(x) \leq a_{\nu+1}\}$. We would like such sets to be measurable, i.e. elements of $\mathcal{F}$. So it seems reasonable that we demand that the pre-images of the intervals $(a_\nu, a_{\nu+1}]$, or the σ-field generated by them, should lie in $\mathcal{F}$.

Definition

A map $f : (X, \mathcal{F}) \to \mathbb{R}^*$ is *$\mathcal{F}$-measurable* if, and only if, $f^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{B}^*$.

Special cases: We say $f : (\mathbb{R}, \mathcal{L}_F) \to \mathbb{R}^*$ is *Lebesgue-Stieltjes measurable* and $f : (\mathbb{R}, \mathcal{B}) \to \mathbb{R}^*$ is *Borel measurable.*

In general we have

Definition A map $f : (X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)$ between measurable spaces is said to be *measurable with respect to* $\mathcal{F}_X, \mathcal{F}_Y$*,* if, and only if, $f^{-1}(A) \in \mathcal{F}_X$ for all $A \in \mathcal{F}_Y$.

(Compare this to the definition of a continuous function between topological spaces.)

***Note** To say that $f : (X, \mathcal{F}) \to \mathbb{R}^*$ is $\mathcal{F}$-measurable is equivalent to saying $f : (X, \mathcal{F}) \to (\mathbb{R}^*, \mathcal{B}^*)$ measurable with respect to $\mathcal{F}, \mathcal{B}^*$. It would be too restrictive to look only at functions $f : (X, \mathcal{F}) \to (\mathbb{R}^*, \mathcal{L}^*)$ measurable with respect to $\mathcal{F}, \mathcal{L}^*$. (The definition of $\mathcal{L}^*$ should be obvious.) It is shown in an appendix that $\mathcal{L}$ is strictly larger than $\mathcal{B}$ and so $\mathcal{L}^*$ is strictly larger than $\mathcal{B}^*$. So if f is measurable with respect to $\mathcal{F}, \mathcal{L}^*$ we are saying that $f^{-1}(V) \in \mathcal{F}$ for all $V \in \mathcal{L}^*$ which is demanding more than $f^{-1}(V) \in \mathcal{F}$ just for all $V \in \mathcal{B}^*$ and so is satisfied by fewer functions.

We want to give a criteria for checking whether a function is $\mathcal{F}$-measurable that is quicker than looking at $f^{-1}(B)$ for all extended Borel sets. First we state results concerning preimages: It can be checked by the student that for any sets $A_i, i \in I$ we have

$$f^{-1}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f^{-1}(A_i), \qquad f^{-1}\left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} f^{-1}(A_i) \tag{12}$$

and $f^{-1}(A^c) = \left(f^{-1}(A)\right)^c$.

Notation If A is a collection of sets then $h^{-1}(\mathcal{A})$ is the collection of preimages of each set in $\mathcal{A}$.

We use this notation and (12) in the "tricky" proof of the following.

Lemma 3.2 *If* $h : (X, \mathcal{F}) \to \mathbb{R}^*$ *and* $\mathcal{A}$ *is a non-empty collection of subsets of* $\mathbb{R}^*$*, then*

$$\sigma(h^{-1}(\mathcal{A})) = h^{-1}(\sigma(\mathcal{A})).$$

Proof We first show that $\sigma(h^{-1}(\mathcal{A})) \subseteq h^{-1}(\sigma(\mathcal{A}))$ by showing that $h^{-1}(\sigma(\mathcal{A}))$ is a σ-field. Let $\{B_i\}_{i\geq 1} \subseteq h^{-1}(\sigma(\mathcal{A}))$ be a countable collection of sets. Then for all i we have that $B_i = h^{-1}(A_i)$ for some $A_i \in \sigma(\mathcal{A})$. Since $\sigma(\mathcal{A})$ is a σ-field we have $\bigcup_{i\geq 1} A_i \in \sigma(\mathcal{A})$ and so

$$
\begin{aligned}
\bigcup_{i\geq 1} B_i &= \bigcup_{i\geq 1}(h^{-1}(A_i)) \\
&= h^{-1}\left(\bigcup_{i\geq 1} A_i\right) && \text{by (12),} \\
&\in h^{-1}(\sigma(\mathcal{A})).
\end{aligned}
$$

Thus $h^{-1}(\sigma(\mathcal{A}))$ is closed under countable unions. Now take any $B, C \in h^{-1}(\sigma(\mathcal{A}))$ so $B = h^{-1}(S)$ and $C = h^{-1}(T)$ for some $S, T \in \sigma(\mathcal{A})$. Then

$$
\begin{aligned}
B \setminus C &= B \cap C^c \\
&= h^{-1}(S) \cap h^{-1}(T^c) && \text{by (12),} \\
&= h^{-1}(S \cap T^c) && \text{by (12)} \\
&= h^{-1}(S \setminus T) \\
&\in h^{-1}(\sigma(\mathcal{A})) \text{ since } \sigma(\mathcal{A}) \text{ is a } \sigma\text{-field.}
\end{aligned}
$$

Hence $h^{-1}(\sigma(\mathcal{A}))$ is **a** σ-field. It obviously contains $h^{-1}(\mathcal{A})$ and so contains **the** minimal σ-field containing $h^{-1}(\mathcal{A})$, that is, $\sigma(h^{-1}(\mathcal{A})) \subseteq h^{-1}(\sigma(\mathcal{A}))$.

To obtain the reverse set inclusion we look at what sets have a preimage in $\sigma(h^{-1}(\mathcal{A}))$, hopefully all the sets in $\sigma(\mathcal{A})$ have a preimage in $\sigma(h^{-1}(\mathcal{A}))$. Consider now $\mathcal{H} = \{E \subseteq \mathbb{R}^* : h^{-1}(E) \in \sigma(h^{-1}(\mathcal{A}))\}$. From (12) we can quickly check that this is a σ-field. It trivially contains $\mathcal{A}$ and so $\sigma(\mathcal{A}) \subseteq \mathcal{H}$. By the definition of $\mathcal{H}$ this means that $h^{-1}(\sigma(\mathcal{A})) \subseteq \sigma(h^{-1}(\mathcal{A}))$.

Hence equality. ■

Theorem 3.3

The function $f : (X, \mathcal{F}) \to \mathbb{R}^*$ is $\mathcal{F}$-measurable if, and only if,

$$\{x : f(x) > c\} \in \mathcal{F}$$

for all $c \in \mathbb{R}$.

Proof Let $\mathcal{A}$ be the collection of semi-infinite intervals $(c, +\infty]$ for all $c \in \mathbb{R}$. Then by Proposition 3.1 we have that $\sigma(\mathcal{A}) = \mathcal{B}^*$. So if we start with the definition of F-measurable we find

$$
\begin{aligned}
f^{-1}(\mathcal{B}^*) \subseteq \mathcal{F} \quad &\text{iff} \quad f^{-1}(\sigma(\mathcal{A})) \subseteq \mathcal{F} \\
&\text{iff} \quad \sigma(f^{-1}(\mathcal{A})) \subseteq \mathcal{F} && \text{by Lemma 3.2,} \\
&\text{iff} \quad f^{-1}(\mathcal{A}) \subseteq \mathcal{F} && \text{since } \mathcal{F} \text{ is a } \sigma\text{-field,} \\
&\text{iff} \quad f^{-1}((c, +\infty]) \subseteq \mathcal{F} && \text{for all } c \in \mathbb{R}, \text{ by definition of } \mathcal{G}, \\
&\text{iff} \quad \{x : f(x) > c\} \in \mathcal{F} && \text{for all } c \in \mathbb{R}.
\end{aligned}
$$

$\blacksquare$

(For students: do check that you understand exactly why, in the proof, we have $\sigma(f^{-1}(\mathcal{A})) \subseteq \mathcal{F}$ iff $\quad f^{-1}(\mathcal{A}) \subseteq \mathcal{F}$ (see Question13).)

Notes (i) Theorem 3.3 is often taken as the definition of $\mathcal{F}$-measurable.

(ii) It is easy to show that f is $\mathcal{F}$-measurable if, and only if,

$$\begin{array}{rlll} & \{x & : & f(x) < c\} \in \mathcal{F} \quad \text{for all } c \in \mathbb{R} \\ \text{or} & \{x & : & f(x) \geq c\} \in \mathcal{F} \quad \text{for all } c \in \mathbb{R} \\ \text{or} & \{x & : & f(x) \leq c\} \in \mathcal{F} \quad \text{for all } c \in \mathbb{R}. \end{array}$$

Example 10

$f : (X, \mathcal{F}) \to \mathbb{R}^*$, $f \equiv \kappa$ a constant, possibly $\pm\infty$, is $\mathcal{F}$-measurable. This is simply because

$$\{x : f(x) > c\} = \begin{cases} X \text{ if } c < \kappa \\ \phi \text{ if } c \geq \kappa. \end{cases}$$

In all cases the resulting set is in $\mathcal{F}$.

The next result supplies us with many examples of measurable functions.

Example 11

Let $g : (\mathbb{R}, \mathcal{L}_F) \to \mathbb{R}$ be $\mathcal{L}_F$-measurable, $f : \mathbb{R} \to \mathbb{R}$ be continuous and $h : (X, \mathcal{F}) \to \mathbb{R}$ be $\mathcal{F}$-measurable. Then

(i) *f is Lebesgue measurable,*

(ii) *the composite $f \circ g$ is $\mathcal{L}_F$-measurable,*

(iii) *the composite $f \circ h$ is $\mathcal{F}$-measurable.*

Proof Note that f and g are finite valued so we need only look at the preimage of $(c, \infty) = \bigcup_{n \geq 1}(c, n) \in \mathcal{U}$, the usual topology on $\mathbb{R}$.

(i) Since f is continuous we have that $f^{-1}((c, \infty)) \in \mathcal{U}$. But $\mathcal{U} \subseteq \mathcal{B} \subseteq \mathcal{L}_F$, and so

$$\{x : f(x) > c\} = f^{-1}((c, \infty)) \in \mathcal{L}_F.$$

(ii) Since $g : (\mathbb{R}, \mathcal{L}_F) \to \mathbb{R}$ is $\mathcal{L}_F$-measurable and $f^{-1}((c, \infty)) \in \mathcal{U} \subseteq \mathcal{B}$ then $g^{-1}\left(f^{-1}((c, \infty))\right) \in \mathcal{L}_F$. Hence $(f \circ g)^{-1}((c, \infty)) \in \mathcal{L}_F$ and so $f \circ g$ is $\mathcal{L}_F$-measurable.

(iii) Since $h : (X, \mathcal{F}) \to \mathbb{R}$ is $\mathcal{F}$-measurable and $f^{-1}((c, \infty)) \in \mathcal{U} \subseteq \mathcal{B}$ then $h^{-1}\left(f^{-1}((c, \infty))\right) \in \mathcal{L}_F$ and so $f \circ h$ is $\mathcal{L}_F$-measurable. $\blacksquare$

Theorem 3.4

Let $f, g : (X, \mathcal{F}) \to \mathbb{R}^*$ be $\mathcal{F}-$measurable functions. Let $\alpha, \beta \in \mathbb{R}$. Then

(i) $f+\alpha$ *and* αf *are* $\mathcal{F}$*-measurable,*

(ii) f^2 *is* $\mathcal{F}$*-measurable,*

(iii) $\{x \in X : f(x) > g(x)\} \in \mathcal{F}$,

(vi) $\{x \in X : f(x) = g(x)\} \in \mathcal{F}$,

(v) *on the set of* x *for which it is defined,* $\alpha f + \beta g$ *is* $\mathcal{F}$*-measurable,*

(vi) fg *is* $\mathcal{F}$*-measurable,*

(vii) *on the set of* x *for which it is defined,* f/g *is* $\mathcal{F}$*-measurable,*

(viii) $\max(f,g)$ *and* $\min(f,g)$ *are* $\mathcal{F}$*-measurable,*

(ix) $|f|$ *is* $\mathcal{F}$*-measurable.*

Proof

(i) $\{x \in X : f(x) + \alpha > c\} = \{x \in X : f(x) > c - \alpha\} \in \mathcal{F}$ since f is $\mathcal{F}$-measurable. Hence $f + \alpha$ is $\mathcal{F}$-measurable.

If $\alpha = 0$ then $\{x \in X : \alpha f(x) > c\} = \phi$ if $c \geq 0$ and X if $c < 0$. In both cases the set is in $\mathcal{F}$.

If $\alpha > 0$ then $\{x \in X : \alpha f(x) > c\} = \{x \in X : f(x) > \frac{c}{\alpha}\} \in \mathcal{F}$ since f is $\mathcal{F}$-measurable.

If $\alpha < 0$ then $\{x \in X : \alpha f(x) > c\} = \{x \in X : f(x) < \frac{c}{\alpha}\} \in \mathcal{F}$ since f is $\mathcal{F}$-measurable.

In all cases $\{x \in X : \alpha f(x) > c\} \in \mathcal{F}$ and so αf is $\mathcal{F}$-measurable.

(ii)

$$\{x \in X : f^2(x) > c\} = \begin{cases} X & \text{if } c < 0 \\ \{x \in X : f(x) > \sqrt{c}\} \cup \{x \in X : f(x) < -\sqrt{c}\} \\ & \text{if } c \geq 0. \end{cases}$$

In all cases the resulting set is in $\mathcal{F}$.

(iii) Note that for any two numbers c, d we have $c > d$ if, and only if, there exists a rational number r such that $c > r > d$. Hence

$$\{x \in X : f(x) > g(x)\} = \bigcup_{r \in \mathbb{Q}} \left(\{x \in X : f(x) > r\} \cap \{x \in X : r > g(x)\}\right).$$

All sets on the right are in $\mathcal{F}$ as is the intersection and countable union.

(iv) As in part (iii) we can show that $\{x \in X : g(x) > f(x)\} \in \mathcal{F}$. Then

$$\begin{aligned} \{x &\in X : f(x) = g(x)\} \\ &= X \setminus (\{x \in X : f(x) > g(x)\} \cup \{x \in X : g(x) > f(x)\}) \end{aligned}$$

is an element of $\mathcal{F}$.

(v) By part (i) it suffices to prove that $f+g$ is $\mathcal{F}$-measurable. Recall that $(+\infty)+(-\infty)$ is not defined so $f+g$ is defined only on $X \setminus A$ where

$$\begin{aligned} A &= \{x \in X : f(x) = \pm\infty, g(x) = \mp\infty\} \\ &= \{x \in X : f(x) = -g(x)\} \cap \{x \in X : f(x) = \pm\infty\} \\ &\in \mathcal{F} \end{aligned}$$

by Example 10 and part (iv). Then

$$\{x \in X \setminus A : f(x) + g(x) > c\} = (X \setminus A) \cap \{x \in X : f(x) > c - g(x)\}$$

which is in $\mathcal{F}$ by (iii).

(vi) Continue with the notation of part (v). Then on $X \setminus A$ we can meaningfully write

$$fg = \frac{(f+g)^2 - f^2 - g^2}{2},$$

which is therefore $\mathcal{F}$-measurable, on $X \setminus A$, by parts (ii) and (v). On A we have either $f(x) = +\infty$ and $g(x) = -\infty$ or $f(x) = -\infty, g(x) = +\infty$. In both cases fg is defined with value $-\infty$. Thus $\{x \in A : fg(x) > c\} = \phi \in \mathcal{F}$ for all $c \in \mathbb{R}$. Hence fg is $\mathcal{F}$-measurable, on X.

(vii) By part (vi) it suffices to prove that $1/g$ is $\mathcal{F}$-measurable. This is only defined on $X \setminus B$ where

$$B = \{x \in X : g(x) = 0\} \in \mathcal{F}$$

by Example 10. First assume $c > 0$ then

$$\begin{aligned} \left\{x \in X \setminus B : \frac{1}{g(x)} > c\right\} &= \left\{x \in X \setminus B : 0 \le g(x) < \frac{1}{c}\right\} \\ &= \left\{x \in X : 0 < g(x) < \frac{1}{c}\right\} \\ &= \left\{x \in X : g(x) < \frac{1}{c}\right\} \setminus \{x \in X : g(x) \le 0\} \end{aligned}$$

which is in $\mathcal{F}$ since g is $\mathcal{F}$-measurable. If $c \le 0$ then

$$\begin{aligned}\left\{x\in X\setminus B:\frac{1}{g(x)}>c\right\} &= \{x\in X\setminus B: 0\le g(x)\}\\ &\quad\cup\left\{x\in X\setminus B: g(x)<\frac{1}{c}\right\}\\ &= \{x\in X: 0<g(x)\}\cup\left\{x\in X: g(x)<\frac{1}{c}\right\}\end{aligned}$$

which is in $\mathcal{F}$.

(viii) For the max and min we need do no more than observe that

$$\begin{aligned}\{x &: \max(f(x),g(x))>c\} = \{x: f(x)>c\}\cup\{x:g(x)>c\}\\ \{x &: \min(f(x),g(x))>c\} = \{x: f(x)>c\}\cap\{x:g(x)>c\}.\end{aligned}$$

(ix) Let $f^+=\max(f,0)$ and $f^-=-\min(f,0)$. (The $-$ve sign is taken so that $f^-\ge 0$.) Then f^+ and f^- are $\mathcal{F}$-measurable by (viii). And so $|f|=f^++f^-$ is $\mathcal{F}$-measurable by part (v). ■

3.1 Sequences of Functions

Let $\{x_n\}\subseteq\mathbb{R}^*$ be a sequence of extended real numbers. We can give an extended definition of limit in the following.

Definition

$\lim_{n\to\infty}x_n=\ell$ with ℓ finite if, and only if, $\forall\varepsilon>0\ \exists N: |x_n-\ell|<\varepsilon$ $\forall n\ge N$.

$\lim_{n\to\infty}x_n=+\infty$ if, and only if, $\forall K>0\ \exists N: x_n>K\ \forall n\ge N$.

$\lim_{n\to\infty}x_n=-\infty$ if, and only if, $\forall K<0\ \exists N: x_n<K\ \forall n\ge N$.

Recall the definition of $\sup_{n\ge1}x_n$ can be given as $\alpha=\sup_{n\ge1}x_n$ if $\alpha\ge x_n$ for all n and given any $\varepsilon>0$ there exists $N\ge1$ such that $\alpha-\varepsilon<x_N\le\alpha$. Of course, implicit in this definition is that $\sup_{n\ge1}x_n$ is finite. We can extend to when $\sup_{n\ge1}x_n=+\infty$ or $-\infty$. Of course, in the first case we do not have to check that $+\infty$ is an upper bound since that is necessarily true and in the second case the demand that $-\infty$ means that $x_n=-\infty$ for all n.

Definition

$\sup_{n\ge1}x_n=+\infty$ if, and only if, $\forall K>0\ \exists N: x_N>K$.

$\sup_{n\ge1}x_n=-\infty$ if, and only if, $\forall n\ge1, x_n=-\infty$.

$\inf_{n\ge1}x_n=+\infty$ if, and only if, $\forall n\ge1, x_n=+\infty$.

$\inf_{n\ge1}x_n=-\infty$ if, and only if $\forall K<0\ \exists N: x_N<K$.

Of course a sequence need not have a limit. But we can define forms of limit that exist for all sequences.

Definition

$$\begin{aligned} \limsup_{n\to\infty} x_n &= \lim_{n\to\infty}\left(\sup_{r\geq n} x_r\right). \\ \liminf_{n\to\infty} x_n &= \lim_{n\to\infty}\left(\inf_{r\geq n} x_r\right). \end{aligned}$$

We can see that these always exist in the following way. We note that $\{x_r\}_{r\geq n+1} \subseteq \{x_r\}_{r\geq n}$ and so

$$\sup_{r\geq n+1} x_r \leq \sup_{r\geq n} x_r.$$

Thus $\{\sup_{r\geq n} x_r\}_{n\geq 1}$ is a decreasing sequence and because of our extended definition of limit such a series converges. Either the sequence is bounded below when it converges to a finite value, namely the infimum of the sequence, or it is not bounded below when it converges to $-\infty$ by the extended definition above, which again is the infimum of the sequence. So in both cases we find that

$$\limsup_{n\to\infty} x_n = \inf_{n\geq 1}\left(\sup_{r\geq n} x_r\right).$$

In the same way we have that $\{\inf_{r\geq n} x_r\}_{n\geq 1}$ is an increasing sequence. This leads to

$$\liminf_{n\to\infty} x_n = \sup_{n\geq 1}\left(\inf_{r\geq n} x_r\right).$$

We then have the important result

Theorem 3.5

The limit $\lim_{n\to\infty} x_n$ *exists if, and only if,* $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$. *The common value (even if* $+\infty$ *or* $-\infty$*) is the value of the limit.*

***Proof** (Not given in lecture.)

($\Rightarrow$) Assume $\lim_{n\to\infty} x_n$ exists and is finite, ℓ say. Then

$$\forall \varepsilon > 0\ \exists N : |x_n - \ell| < \varepsilon \qquad \forall n \geq N$$

and so $\ell - \varepsilon < x_n < \ell + \varepsilon$ for such n. That is, $\ell - \varepsilon$ is a lower bound for $\{x_r\}_{r\geq n}$ for any $n \geq N$ and thus $\ell - \varepsilon \leq \inf\{x_r\}_{r\geq n} = \inf_{r\geq n} x_r$. But also, $\inf_{r\geq n} x_r \leq x_n < \ell + \varepsilon$. So

$$\ell - \varepsilon \le \inf_{r \ge n} x_r < \ell + \varepsilon \qquad \forall n \ge N.$$

(Note how a strict inequality has changed to a $\le$). This shows that the definition of limit is satisfied for the sequence $\{\inf_{r\ge n} x_r\}_{n\ge 1}$ and so $\lim_{n\to\infty}(\inf_{r\ge n} x_r) = \ell$. Similarly

$$\ell - \varepsilon < \sup_{r \ge n} x_r \le \ell + \varepsilon \qquad \forall n \ge N$$

leading to $\lim_{n\to\infty}(\sup_{r\ge n} x_r) = \ell$.

Assume $\lim_{n\to\infty} x_n$ exists and is $+\infty$. Then

$$\forall K > 0 \ \exists N : x_n > K \qquad \forall n \ge N.$$

In particular $K \le \inf_{r\ge n} x_r$ for such n. (Obviously $\sup_{r\ge n} x_r = +\infty$ and so $\lim_{n\to\infty}(\sup_{r\ge n} x_r) = +\infty$) But now we have seen that the extended definition of limit is satisfied for the sequence $\{\inf_{r\ge n} x_r\}_n$ and so $\lim_{n\to\infty}(\inf_{r\ge n} x_r) = +\infty$.

The same proof holds when $\lim_{n\to\infty} x_n$ exists and is $-\infty$.

($\Leftarrow$) Assume now that $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$ with a finite limit, ℓ, say. Then $\liminf_{n\to\infty} x_n = \ell$ means that

$$\forall \varepsilon > 0 \ \exists N_1 : \left| \inf_{r\ge n} x_r - \ell \right| < \varepsilon \ \forall n \ge N_1.$$

In particular

$$x_n \ge \inf_{r\ge n} x_r > \ell - \varepsilon \tag{13}$$

for such n.

Similarly $\limsup_{n\to\infty} x_n = \ell$ means that

$$\forall \varepsilon > 0 \ \exists N_2 : \left| \sup_{r\ge n} x_r - \ell \right| < \varepsilon \ \forall n \ge N_2.$$

In particular

$$x_n \le \sup_{r\ge n} x_r < \ell + \varepsilon \tag{14}$$

for such n. Let $N = \max(N_1, N_2)$ then for all $n \ge N$ we can combine (13) and (14) to get $\ell - \varepsilon < x_n < \ell + \varepsilon$ and so $\lim_{n\to\infty} x_n = \ell$.

Assume now that $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = +\infty$. Then

$$\forall K > 0 \ \exists N : \inf_{r \geq n} x_r > K \ \forall n \geq N$$

and in particular $x_n \geq \inf_{r \geq n} x_r > K \ \forall n \geq N$ and so $\lim_{n\to\infty} x_n = +\infty$.

A similar proof holds when the common limit is $-\infty$. ■

Note For any sequence $\{x_n\} \subseteq \mathbb{R}^*$ we have $\sup_{n\geq 1} x_n = -\inf_{n\geq 1}(-x_n)$ and $\sup_{n\geq 1} x_n > c$ if, and only if, there exists $i : x_i > c$.

(*Proof is left to student. For the second result note first that $\sup_{n\geq 1} x_n = +\infty$ iff $\forall K > 0 \ \exists N : x_N > K$. Choose $K = c$ to get the result. Otherwise $\sup_{n\geq 1} x_n = \ell$ a finite value when we know that given any $\varepsilon > 0$ there exists $N \geq 1$ such that $\ell - \varepsilon < x_N \leq \ell$. Simply choose ε so that $\ell - \varepsilon \geq c$, perhaps $\varepsilon = (\ell - c)/2$, to get the result.).

Let $f_n : (X, \mathcal{F}) \to \mathbb{R}^*$ be a sequence of $\mathcal{F}$-measurable functions and define $\sup_{n\geq 1} f_n$ and $\inf_{n\geq 1} f_n$ pointwise, that is, for all $x \in X$ define $(\sup_{n\geq 1} f_n)(x) = \sup_{n\geq 1} f_n(x)$ and $(\inf_{n\geq 1} f_n)(x) = \inf_{n\geq 1} f_n(x)$.

Theorem 3.6

i) *The functions* $\sup_{n\geq 1} f_n$ *and* $\inf_{n\geq 1} f_n$ *are* $\mathcal{F}$*-measurable functions.*

ii) *The functions* $\liminf_{n\to\infty} f_n$ *and* $\limsup_{n\to\infty} f_n$ *are* $\mathcal{F}$*-measurable functions.*

iii) *The set of* $x \in X$ *for which* $\lim_{n\to\infty} f_n(x)$ *exists is a measurable set.*

iv) *On the set of* x *for which* $\lim_{n\to\infty} f_n(x)$ *exists the limit function is* $\mathcal{F}$*-measurable.*

Proof

i) Let $c \in \mathbb{R}$. From the note above we have

$$\begin{aligned}
\left\{ x : \sup_{n\geq 1} f_n(x) > c \right\} &= \{x : \text{ there exists } i \text{ for which } f_i(x) > c\} \\
&= \bigcup_{i\geq 1} \{x : f_i(x) > c\} \in \mathcal{F}
\end{aligned}$$

since each f_i is $\mathcal{F}$-measurable and $\mathcal{F}$ is closed under countable unions. Hence $\sup_{n\geq 1} f_n$ is $\mathcal{F}$-measurable.

For the infimum we use the note again to deduce that $\inf_{n\geq 1} f_n = -\sup_{n\geq 1} (-f_n)$ is $\mathcal{F}$-measurable.

ii) As observed above we have

$$\liminf_{n\to\infty} f_n = \sup_{n\geq 1} \left(\inf_{r\geq n} f_r \right) \quad \text{and} \quad \limsup_{n\to\infty} f_n = \inf_{n\geq 1} \left(\sup_{r\geq n} f_r \right).$$

So part (i) gives the result for $\liminf\limits_{n\to\infty} f_n$ and $\limsup\limits_{n\to\infty} f_n$.

iii) By Theorem 3.5 we have

$$\left\{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists}\right\} = \left\{x \in X : \liminf_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x)\right\}.$$

So our set is that of points at which two $\mathcal{F}$-measurable functions are equal. By Theorem 3.4(vi) such a set is an element of $\mathcal{F}$.

iv) Let $A = \{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists}\}$ then

$$\begin{aligned}
& \left\{x \in A : \lim_{n\to\infty} f_n(x) > c\right\} \\
= & \left\{x \in A : \liminf_{n\to\infty} f_n(x) > c\right\} \qquad \text{since } \liminf_{n\to\infty} f_n = \lim_{n\to\infty} f_n \text{ on } A, \\
= & \; A \cap \left\{x \in X : \liminf_{n\to\infty} f_n(x) > c\right\} \qquad \text{since } \liminf_{n\to\infty} f_n \text{ defined on } \textbf{all} \text{ of } X, \\
\in & \; \mathcal{F},
\end{aligned}$$

using parts (ii) and (iii).

■

Note This limit result for measurable functions does not necessarily hold for continuous functions even though continuous functions are measurable. For example, $f_n(x) = x^n$ are continuous on $[0, 1]$ yet $\inf f_n(x) = 0$ for $0 \leq x < 1$ and 1 when $x = 1$, and so not continuous.