2.6 Lebesgue measurable sets.

For a set $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$ define the translated set $A(x) = A + x = \{y + x : y \in A\}$.

Lemma 2.13 (Invariance under translation)†

Let $E \in \mathcal{L}$ and $x \in \mathbb{R}$ then $E(x) \in \mathcal{L}$ and $\mu(E) = \mu(E(x))$.

Proof

(i) If $I \in \mathcal{P}$ then $I = (a, b]$ for some a and b and

$$\begin{aligned} \mu(I(x)) &= \mu((a + x, b + x]) \\ &= (b + x) - (a + x) \\ &= b - a = \mu(I). \end{aligned}$$

(ii) If $E \in \mathcal{E}$ then $E = \bigcup_{i=1}^n I_i$, a disjoint union of $I_i \in \mathcal{P}$. By definition of the extended measure given in the proof of Theorem 2.2 we have $\mu(E) = \sum_{i=1}^n \mu(I_i)$ which by part (i) is translation invariant.

(iii) We now look at the outer measure μ^*. Let $A \subseteq \mathbb{R}$. Then there is a map between the covers $\{E_i\}_{i\geq 1} \subseteq \mathcal{E}$ of A and the covers $\{E_i'\}_{i\geq 1} \subseteq \mathcal{E}$ of $A(x)$ given by $E_i \to E_i(x)$ and $E_i' \to E_i'(-x)$. By (ii) we have that $\mu(E_i) = \mu(E_i(x))$ and $\mu(E_i') = \mu(E_i'(x))$ and so the infimum of the values $\sum_{i=1}^\infty \mu(E_i)$ is the same as for the values $\sum_{i=1}^\infty \mu(E_i')$. Hence $\mu^*(A) = \mu^*(A(x))$ and so μ^* is translation invariant.

(iv) Let $E \in \mathcal{L}$ and $x \in \mathbb{R}$ be given. Take any test set $A \subseteq \mathbb{R}$. Apply the definition of measurable set to E with test set $A(-x)$ to get

$$\mu^*(A(-x)) = \mu^*(A(-x) \cap E) + \mu^*(A(-x) \cap E^c). \tag{11}$$

But

$$\begin{aligned} y \in A(-x) \cap E \quad &\text{iff } y \in A(-x) \text{ and } y \in E \\ &\text{iff } y + x \in A \text{ and } y + x \in E(x) \\ &\text{iff } y + x \in A \cap E(x) \\ &\text{iff } y \in (A \cap E(x))(-x). \end{aligned}$$

So $A(-x) \cap E = (A \cap E(x))(-x)$. Similarly $A(-x) \cap E^c = (A \cap E(x)^c)(-x)$. Since μ^* is translation invariant (11) becomes

$$\mu^*(A) = \mu^*(A \cap E(x)) + \mu^*(A \cap E(x)^c).$$

True for all $A \subseteq X$ means $E(x) \in \mathcal{L}$. ■

*2.7 Non-measurable sets

Theorem 2.14

There exists a subset $V \subseteq \mathbb{R}$ for which $\mu(V)$ is not defined. That is, there exist non-measurable sets.

***Proof** Define $\mathbb{Q}_1 = \mathbb{Q} \cap [-1,1]$. Given $x, y \in [0,1]$ define $x \sim y$ if, and only if, $x - y \in \mathbb{Q}_1$. This is an equivalence relation and splits $[0,1]$ into a disjoint union $[0,1] = \bigcup_\alpha E_\alpha$ of equivalence classes. If $x \in E_\alpha$ then each $y \in E_\alpha$ satisfies $y - x \in \mathbb{Q}_1$. Then, since $\mathbb{Q}_1$ is countable, so are the number of y, that is E_α is countable. Yet $[0,1]$ is uncountable so the collection of E_α is uncountable.

Choose one element from each of the E_α and collect together to form a set V and if necessary relabel the E_α so that $\alpha \in V$.$^\diamond$

Assume that V is Lebesgue measurable.

Enumerate $\mathbb{Q}_1 = \{r_1, r_2, r_3, ...\}$ and translate V by r_n, denoting each resulting set by $V_n = V(r_n)$. These are disjoint sets. For if $V_n \cap V_m \neq \phi$ choose some $y \in V_n \cap V_m$. Then $y - r_n, y - r_m \in V$, but $(y - r_n) - (y - r_m) = r_m - r_n \in \mathbb{Q}_1$ and so $y - r_n$ and $y - r_m$ lie in the same equivalence class in $[0,1]$. But V contains only one point from each class hence $y - r_n = y - r_m$ and, thus, $V_n = V_m$.

Given any $x \in [0,1]$ then $x \in E_\alpha$ for some $\alpha \in V$. So $x = \alpha + r_n$ for some $r_n \in \mathbb{Q}_1$, that is

$$x = \alpha + r_n \in V + r_n = V(r_n) = V_n.$$

Thus $[0,1] \subseteq \bigcup_{i=1}^\infty V_i$. Also $\bigcup_{i=1}^\infty V_i \subseteq [-1,2]$. Then

$$3 = \mu([-1,2]) \geq \mu\left(\bigcup_{i=1}^\infty V_i\right) \geq \sum_{i=1}^\infty \mu(V_i) = \sum_{i=1}^\infty \mu(V)$$

since μ is translation invariant. Hence we must have $\mu(V) = 0$. But

$$1 = \mu([0,1]) \leq \mu\left(\bigcup_{i=1}^\infty V_i\right) \leq \sum_{i=1}^\infty \mu(V_i) = \sum_{i=1}^\infty \mu(V).$$

Contradiction, so our assumption is false. Hence V is not Lebesgue measurable. ■

$^\diamond$*There is a very subtle point here concerning the possibility of choosing an element from each set in an *uncountable* collection. If you cannot list the sets how can you ensure that you have visited each set to choose an element. In fact it has to be postulated that you can do this and it is called the *Axiom of Choice*. It is known that no contradictions arise in mathematics if you assume this axiom but it is also known that no contradictions arise if you do

not assume it. So, strangely, it is not known if the axiom of choice is true. Most mathematicians assume the axiom, though it does lead to proofs that objects exist without being able to construct them. For instance, if we don't assume the axiom of choice we cannot find a non-measurable set, i.e. we only "know" that a non-measurable set exists if we assume the axiom of choice.

Corollary 2.15†

The outer measure μ_F^* on $\mathbb{R}$ is not a measure on $\mathbb{R}$.

Proof.

Let T be a non-measurable set, so T does not satisfy (6) for some test set $A \subseteq \mathbb{R}$. That is,

$$\mu_F^*(A) \neq \mu_F^*(A \cap T) + \mu_F^*(A \cap T^c).$$

So μ_F^* is not additive and so not a measure. ■

2.8 Sets of measure zero.

Example 8 All countable subsets of $\mathbb{R}$ have Lebesgue measure 0.

Verification Given a point $a \in \mathbb{R}$ we have $\{a\} = \bigcap_{n\geq 1}(a - 1/n, a] \in \mathcal{L}$, so $\{a\}$ is Lebesgue measurable. It's measure satisfies $\mu\{a\} \leq \mu(a-1/n, a] = 1/n$ for all $n \geq 1$. Hence $\mu\{a\} = 0$. So, for a countable set A we have

$$\mu(A) = \mu\left(\bigcup_{a\in A}\{a\}\right) = \sum_{a\in A}\mu(A) = 0.$$

Below we will give construct non-countable sets with measure zero.

Definition Given a measure $\mu : \mathcal{C} \to \mathbb{R}^+$, the class $\mathcal{C}$ is *complete with respect to* μ if $E \subseteq F, F \in \mathcal{C}$ and $\mu(F) = 0$ altogether imply $E \in \mathcal{C}$. We also say the μ is a *complete measure.*

Lemma 2.16

If μ is obtained by restricting an outer measure μ^* to the class $\mathcal{M}$ of μ^*-measurable sets then μ is complete.

Proof Let $A \subseteq X$ be given. If $E \subseteq F$ and $\mu^*(F) = 0$ then $\mu^*(E) = 0$ and $\mu^*(A \cap E) = 0$ since μ^* is monotone. So

$$\begin{aligned} \mu^*(A) &\geq \mu^*(A \cap E^c) && \text{since } \mu^* \text{ is monotone} \\ &= \mu^*(A \cap E^c) + \mu^*(A \cap E). \end{aligned}$$

Thus (7) is satisfied and so $E \in \mathcal{M}$. ■

Corollary 2.17†

The Lebesgue-Stieltjes measure on $\mathbb{R}$ is complete.

■

Example 9 Cantor Set

The Cantor set K is constructed by defining a sequence of sets

$$\begin{aligned} F_0 &= [0,1] \\ F_1 &= [0,1/3] \cup [2/3,1] \\ F_2 &= [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1] \\ &\vdots \end{aligned}$$

So F_n is constructed by removing the middle open third intervals from F_{n-1}. The Cantor set is $K = \bigcap_{n=1}^{\infty} F_n$.

Recall from Question 10, that the Borel sets $\mathcal{B}$ are generated by the closed intervals $[a,b]$. Also recall that a σ-field is closed under countable intersections. Hence K is a Borel set. This means that $K \subseteq \mathcal{B} \subseteq \mathcal{L}_F$ for any F. We just restrict to $K \subseteq \mathcal{L}$.

At first sight K might be seen to contain only points such as $\frac{1}{3}, \frac{1}{9}, \frac{2}{3}, \frac{1}{27}, \ldots$, and this is supported when we look at the Lebesgue measure of K. For each n we have

$$\begin{aligned} \mu(K) &\leq \mu(F_n) \qquad \text{since } \mu \text{ is monotone} \\ &= \frac{2}{3}\mu(F_{n-1}) = \ldots \\ &= \left(\frac{2}{3}\right)^n \mu(F_0) = \left(\frac{2}{3}\right)^n . \end{aligned}$$

True for all n and so $\mu(K) = 0$. So in the sense of measure the Cantor set is "small".

Alternatively look at the numbers in $[0,1]$ in a ternary expansion, so

$$x = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \frac{a_4}{3^4} + \frac{a_5}{3^5} + \ldots ,$$

where each $a_i = 0, 1$ or 2. Take the convention that if the last non-zero coefficient is 1 then we choose the non-terminating expansion. So, for example, for $\frac{1}{3}$ we take the expansion

$$\frac{0}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \frac{2}{3^4} + \frac{2}{3^5} + \ldots ,$$

while for $8/9$ we take

$$\frac{2}{3}+\frac{2}{3^2}+\frac{0}{3^3}+\frac{0}{3^4}+\frac{0}{3^5}+\dots,$$

or for 7/9 we take

$$\frac{2}{3}+\frac{0}{3^2}+\frac{2}{3^3}+\frac{2}{3^4}+\frac{2}{3^5}+\dots.$$

Then

$$\begin{aligned}
x &\in F_1 \text{ iff } a_1 = 0 \text{ or } 2\\
x &\in F_2 \text{ iff } (a_1 = 0 \text{ or } 2) \text{ and } (a_2 = 0 \text{ or } 2)\\
x &\in F_3 \text{ iff } (a_1 = 0 \text{ or } 2) \text{ and } (a_2 = 0 \text{ or } 2) \text{ and } (a_3 = 0 \text{ or } 2)\\
&\vdots
\end{aligned}$$

So $x \in K$ if, and only if, $a_i = 0$ or 2 for all $i \geq 1$. (Note, for example, that this gives us, subject to the convention on infinite expansions, that $\frac{1}{3} \in K$ as we would expect from the definition of K as an intersection of F_n.)

We can now map K to $[0,1]$ by the map

$$f : \sum_{n=1}^{\infty} \frac{a_n}{3^n} \mapsto \sum_{n=1}^{\infty} \frac{b_n}{2^n}$$

where $b_n = 0$ if $a_n = 0$ and $b_n = 1$ if $a_n = 2$. This map is **not** 1-1. We see this by an example

$$\begin{aligned}
f\left(\frac{8}{9}\right) &= f\left(\frac{2}{3}+\frac{2}{3^2}+\frac{0}{3^3}+\frac{0}{3^4}+\frac{0}{3^5}+\dots\right)\\
&= \frac{1}{2}+\frac{1}{2^2}+\frac{0}{2^3}+\frac{0}{2^4}+\frac{0}{2^5}+\dots = \frac{3}{4},
\end{aligned}$$

while

$$\begin{aligned}
f\left(\frac{7}{9}\right) &= f\left(\frac{2}{3}+\frac{0}{3^2}+\frac{2}{3^3}+\frac{2}{3^4}+\frac{2}{3^5}+\dots\right)\\
&= \frac{1}{2}+\frac{0}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\dots = \frac{3}{4}.
\end{aligned}$$

But the map from $[0,1]$, with the numbers expressed in the unique non-terminating binary form, to K given by

$$h : \sum_{n=1}^{\infty} \frac{b_n}{2^n} \mapsto \sum_{n=1}^{\infty} \frac{2b_n}{3^n},$$

is a 1-1 map into a **subset** of K. For instance, $h(3/4) = 7/9$ while $h(x) = 8/9$ has no solution. In particular K has cardinality at least as large as $[0, 1]$. But since K is a subset of $[0, 1]$ it must have the same cardinality. Yet the cardinality of $[0, 1]$ is the same as $\mathbb{R}$ and denoted by c. In particular K is uncountable. So in the sense of countability the Cantor set is "large".

Corollary 2.18

The cardinality of the collection of Lebesgue measurable sets is 2^c.

Proof

By Lemma 2.16, since $K \in \mathcal{L}$ and $\mu(K) = 0$ any subset $A \subseteq K$ satisfies $A \in \mathcal{L}$.

The number of subsets of K is 2^c hence the cardinality of $\mathcal{L}$ is $\geq 2^c$. But every Lebesgue measurable set is a subset of $\mathbb{R}$, of which there are 2^c. Hence we get the result. ■