2.5 Outer Measure and Measurable sets.

Note The results of this section concern any given outer measure λ.

If an outer measure λ on a set X were a measure then it would be additive. In particular, given any two sets $A, B \subseteq X$ we have that $A \cap B$ and $A \cap B^c$ are disjoint with $(A \cap B) \cup (A \cap B^c) = A$ and so we would have

$$\lambda(A) = \lambda(A \cap B) + \lambda(A \cap B^c).$$

We will see later that this does not necessarily hold for all A and B but it does lead to the following definition.

Definition Let λ be an outer measure on a set X. Then $E \subseteq X$ is said to be *measurable with respect to* λ (or λ*-measurable*) if

$$\lambda(A) = \lambda(A \cap E) + \lambda(A \cap E^c) \qquad \text{for all} \qquad A \subseteq X. \tag{7}$$

(This can be read as saying that we take each and every possible "test set", A, look at the measures of the parts of A that fall within and without E, and check whether these measures add up to that of A.)

Since λ is subadditive we have $\lambda(A) \leq \lambda(A \cap E) + \lambda(A \cap E^c)$ so, in checking measurability, we need only verify that

$$\lambda(A) \geq \lambda(A \cap E) + \lambda(A \cap E^c) \qquad \text{for all} \qquad A \subseteq X. \tag{8}$$

Let $\mathcal{M} = \mathcal{M}(\lambda)$ denote the collection of λ-measurable sets.

Theorem 2.6

$\mathcal{M}$ is a field.

Proof

Trivially ϕ and X are in $\mathcal{M}$.

Take any $E_1, E_2 \in \mathcal{M}$ and any test set $A \subseteq X$. Then

$$\lambda(A) = \lambda(A \cap E_1) + \lambda(A \cap E_1^c).$$

Now apply the definition of measurability for E_2 with the test set $A \cap E_1^c$ to get

$$\begin{aligned} \lambda(A \cap E_1^c) &= \lambda((A \cap E_1^c) \cap E_2) + \lambda((A \cap E_1^c) \cap E_2^c) \\ &= \lambda(A \cap E_1^c \cap E_2) + \lambda(A \cap (E_1 \cup E_2)^c). \end{aligned}$$

Combining

$$\lambda(A) = \lambda(A \cap E_1) + \lambda(A \cap E_1^c \cap E_2) + \lambda(A \cap (E_1 \cup E_2)^c). \tag{9}$$

We hope to use the subadditivity of λ on the first two term on the right hand side of (9). For the sets there we have

$$\begin{aligned}(A \cap E_1) \cup (A \cap E_1^c \cap E_2) &= A \cap (E_1 \cup (E_1^c \cap E_2)) \\ &= A \cap ((E_1 \cup E_1^c) \cap (E_1 \cup E_2)) \\ &= A \cap (X \cap (E_1 \cup E_2)) \\ &= A \cap (E_1 \cup E_2).\end{aligned}$$

So

$$\lambda(A \cap E_1) + \lambda(A \cap E_1^c \cap E_2) \geq \lambda(A \cap (E_1 \cup E_2)).$$

Substituting in (9) gives

$$\lambda(A) \geq \lambda(A \cap (E_1 \cup E_2)) + \lambda(A \cap (E_1 \cup E_2)^c).$$

So we have verified (8) for $E_1 \cup E_2$, that is, $E_1 \cup E_2 \in \mathcal{M}$.

Observe that the definition of λ-measurable sets is symmetric in that $E \in \mathcal{M}$ if, and only if, $E^c \in \mathcal{M}$. Thus

$$E_1 \setminus E_2 = E_1 \cap E_2^c = (E_1^c \cup E_2)^c \in \mathcal{M}.$$

Hence $\mathcal{M}$ is a field. ■

Proposition 2.7 *If $G, F \in \mathcal{M}(\lambda)$ are disjoint then*

$$\lambda(A \cap (G \cup F)) = \lambda(A \cap G) + \lambda(A \cap F)$$

for all $A \subseteq X$.

Proof

Let $A \subseteq X$ be given. Apply the definition of λ-measurability to G with the test set $A \cap (G \cup F)$. Then

$$\begin{aligned}\lambda(A \cap (G \cup F)) &= \lambda((A \cap (G \cup F)) \cap G) \\ &\quad + \lambda((A \cap (G \cup F)) \cap G^c).\end{aligned} \tag{10}$$

Yet

$$\begin{aligned}(G \cup F) \cap G &= (G \cap G) \cup (F \cap G) \\ &= G \cup (F \cap G) = G\end{aligned}$$

since $F \cap G \subseteq G$. Also

$$\begin{aligned} (G \cup F) \cap G^c &= (G \cap G^c) \cup (F \cap G^c) \\ &= \phi \cup F = F, \end{aligned}$$

because F and G disjoint means $F \subseteq G^c$ and so $F \cap G^c = F$. Thus (9) becomes

$$\lambda(A \cap (G \cup F)) = \lambda(A \cap G) + \lambda(A \cap F).$$

■

Using induction it is possible to prove the following.

Corollary 2.8 *For all $n \geq 1$ if $\{F_i\}_{1 \leq i \leq n}$ is a finite collection of disjoint sets from $\mathcal{M}(\lambda)$ then*

$$\lambda\left(A \cap \bigcup_{i=1}^{n} F_i\right) = \sum_{i=1}^{n} \lambda(A \cap F_i) \quad \textit{for all } A \subseteq X.$$

Proof Left to students. ■

Corollary 2.9 *If $\{F_i\}_{i \geq 1}$ is a countable collection of disjoint sets from $\mathcal{M}(\lambda)$ then*

$$\lambda\left(A \cap \bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} \lambda(A \cap F_i) \quad \textit{for all } A \subseteq X.$$

Proof

For any $n \geq 1$ we have $A \cap \bigcup_{i=1}^{\infty} F_i \supseteq A \cap \bigcup_{i=1}^{n} F_i$ and so by monotonicity we have

$$\begin{aligned} \lambda\left(A \cap \bigcup_{i=1}^{\infty} F_i\right) &\geq \lambda\left(A \cap \bigcup_{i=1}^{n} F_i\right) \\ &= \sum_{i=1}^{n} \lambda(A \cap F_i) \end{aligned}$$

by Corollary 2.8. Let $n \to \infty$ to get

$$\lambda\left(A \cap \bigcup_{i=1}^{\infty} F_i\right) \geq \sum_{i=1}^{\infty} \lambda(A \cap F_i).$$

The reverse inequality follows from subadditivity. ■

Theorem 2.10

$\mathcal{M}(\lambda)$ is a σ-field and λ restricted to $\mathcal{M}(\lambda)$ is a measure.

Proof

Let $\{E_i\}_{i\geq 1}$ be a countable collection from $\mathcal{M}$. They might not be disjoint so define $F_1 = E_1$ and

$$F_i = E_i \setminus \bigcup_{j=1}^{i-1} E_j$$

for all $i > 1$. The F_i are disjoint and $F_i \in \mathcal{M}$ since $\mathcal{M}$ is a field. Let $G_m = \bigcup_{j=1}^{m} F_j$ and $G = \bigcup_{j=1}^{\infty} F_j = \bigcup_{i=1}^{\infty} E_i$. Then for any $A \subseteq X$ and for any $n \geq 1$ we have

$$\begin{aligned} \lambda(A) &= \lambda(A \cap G_n) + \lambda(A \cap G_n^c) && \text{since } G_n \in \mathcal{M} \\ &= \sum_{i=1}^{n} \lambda(A \cap F_i) + \lambda(A \cap G_n^c) && \text{by Corollary 2.8} \\ &\geq \sum_{i=1}^{n} \lambda(A \cap F_i) + \lambda(A \cap G^c) && \text{since } G^c \subseteq G_n^c. \end{aligned}$$

True for all n means that

$$\begin{aligned} \lambda(A) &\geq \sum_{i=1}^{\infty} \lambda(A \cap F_i) + \lambda(A \cap G^c) \\ &= \lambda\left(A \cap \bigcup_{i=1}^{\infty} F_i\right) + \lambda(A \cap G^c) \\ &\qquad \text{by Corollary 2.8, since the } F_i \text{ are disjoint,} \\ &= \lambda\left(A \cap G\right) + \lambda(A \cap G^c). \end{aligned}$$

True for all $A \subseteq X$ means that $G \in \mathcal{M}(\lambda)$.

Choosing $A = X$ in Corollary 2.9 shows that λ is σ-additive on $\mathcal{M}(\lambda)$. Hence λ is a measure on $\mathcal{M}(\lambda)$. ■

Example † With the Lebesgue-Stieltjes outer measure μ_F^* of example 8 we can now form the σ-field $\mathcal{M}(\mu_F^*)$. This is known as the collection of *Lebesgue-Stieltjes measurable sets* and is denoted by $\mathcal{L}_F$. If $F(x) = x$, it is simply known as the collection of *Lebesgue measurable sets* and is denoted by $\mathcal{L}$.

We now specialise to those outer measures constructed, as in (4), from measures defined in a ring.

Theorem 2.11 *Let $\mathcal{R}$ be a ring of sets in X such that $X = \bigcup_{i=1}^{\infty} E_i$ for some $E_i \in \mathcal{R}$. Let μ be a measure on $\mathcal{R}$ and let μ^* be the outer measure on X constructed from μ as in (4). Then*

(i) *the elements of $\mathcal{R}$ are μ^*-measurable sets,*

(ii) *$\mu^* = \mu$ on $\mathcal{R}$.*

Proof

(i) Let $E \in \mathcal{R}$ and a test set $A \subseteq X$ be given. If $\mu^*(A) = +\infty$ then (7) is trivially satisfied so assume that $\mu^*(A) < +\infty$.

Let $\varepsilon > 0$ be given. By the definition (4) there exists a countable collection $\{E_i\}_{i\geq 1} \subseteq \mathcal{R}$ such that $A \subseteq \bigcup_{i\geq 1} E_i$ and

$$\mu^*(A) \leq \sum_{i=1}^{\infty} \mu(E_i) < \mu^*(A) + \varepsilon.$$

Yet μ is a measure on $\mathcal{R}$ and $E_i, E \in \mathcal{R}$ so

$$\mu(E_i) = \mu(E_i \cap E) + \mu(E_i \cap E^c).$$

Combining we see

$$\begin{aligned} \mu^*(A) + \varepsilon &> \sum_{i=1}^{\infty} \left(\mu(E_i \cap E) + \mu(E_i \cap E^c)\right) \\ &\geq \mu^*(A \cap E) + \mu^*(A \cap E^c), \end{aligned}$$

since $\{E_i \cap E\}_{i\geq 1}$ and $\{E_i \cap E^c\}_{i\geq 1}$ are covers for $A \cap E$ and $A \cap E^c$ respectively. True for all $\varepsilon > 0$ means that (7) is satisfied and so $E \in \mathcal{M}$ and thus $\mathcal{R} \subseteq \mathcal{M}$.

(ii) Let $E \in \mathcal{R}$ be given. Then since E is a cover from $\mathcal{R}$ for E we have that

$$\mu^*(E) = \inf_{\text{all covers}} \sum \mu(E_i) \leq \mu(E).$$

Take any other cover $\{E_i\}_{i\geq 1}$ of E. As in Theorem 2.10 replace the E_i by disjoint sets $F_i \subseteq E_i, F_i \in \mathcal{R}$ and where $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$. Then

$$
\begin{aligned}
\mu(E) &= \mu\left(E \cap \bigcup_{i=1}^{\infty} F_i\right) && \text{since } E \subseteq \bigcup_{i=1}^{\infty} F_i, \\
&= \mu\left(\bigcup_{i=1}^{\infty} (E \cap F_i)\right) \\
&= \sum_{i=1}^{\infty} \mu\left(E \cap F_i\right) && \text{since } \mu \text{ is additive on } \mathcal{R}, \\
&\leq \sum_{i=1}^{\infty} \mu\left(F_i\right) \leq \sum_{i=1}^{\infty} \mu\left(E_i\right)
\end{aligned}
$$

since $E \cap F_i \subseteq F_i \subseteq E_i$ and μ is monotonic on $\mathcal{R}$. So $\mu(E)$ is a lower bound for the sums for which $\mu^*(E)$ is the greatest lower bound and thus $\mu(E) \leq \mu^*(E)$.

Hence $\mu(E) = \mu^*(E)$. ■

We now see that it is reasonable to say that μ^* *extends* μ.

Further, let $\mathcal{A}$ be a collection of subsets of X.

Definition We say that the extended real-valued function $\phi : \mathcal{A} \to \mathbb{R}^*$ is *σ-finite* if for all $A \in \mathcal{A}$ there exists a countable collection $\{A_n\}_{n\geq 1} \subseteq \mathcal{A}$ such that $A \subseteq \bigcup_{n\geq 1} A_n$ and $|\phi(A_n)| < \infty$ for all $n \geq 1$.

Then it can be shown that

Theorem 2.12

If, in addition to the conditions of Theorem 2.11, μ is σ-finite on $\mathcal{R}$ then the extension to $\mathcal{M}$ is unique and is also σ-finite.

Proof not given here. ■

Example 8† Recall that μ_F was first defined on $\mathcal{P}$ and then extended to $\mathcal{E}$ in Corollary 2.3. In the last example the σ-field $M\left(\mu_F^*\right)$, known as $\mathcal{L}_F$, was constructed. Now Theorem 2.11 implies that $\mathcal{E} \subseteq \mathcal{L}_F$ and $\mu_F^* = \mu_F$ on $\mathcal{E}$. So it is reasonable to write μ_F^* simply as μ_F on $\mathcal{L}_F$, called the *Lebesgue-Stieltjes measure.* If $F(x) = x$, we write μ_F simply as μ, called the *Lebesgue measure* on $\mathbb{R}$.

Note† $\mathcal{L}_F \supseteq \mathcal{E} \supseteq \mathcal{P}$ so $\mathcal{L}_F$ is **a** σ-field containing $\mathcal{P}$. But $\mathcal{B}$, the Borel sets of $\mathbb{R}$, is **the** smallest σ-field containing $\mathcal{P}$. Hence $\mathcal{L}_F \supseteq \mathcal{B}$, true for **all** distribution functions F.