Measure Theory

1 Classes of Subsets

1.1 Topology

Definition A *Topological space* $(X, \mathcal{T})$ consists of a non-empty set X together with a collection $\mathcal{T}$ of subsets of X such that

(T1) $X, \phi \in \mathcal{T}$,
(T2) If $A_1, ..., A_n \in \mathcal{T}$ then $\bigcap_{i=1}^n A_i \in \mathcal{T}$,
(T3) If $A_i \in \mathcal{T}, i \in I$ for some index set I then $\bigcup_{i \in I} A_i \in \mathcal{T}$.

The elements of $\mathcal{T}$ are called the *open sets*.

Note Every set has at least two topologies on it, namely the trivial ones $\mathcal{T} = \{X, \phi\}$ and $\mathcal{T} = \mathcal{P}(X)$, i.e. the collection of *all* subsets of X.

Lemma 1.1 If $\mathcal{T}_j, j \in J$ are topologies on X then $\bigcap_{j \in J} \mathcal{T}_j$ is a topology on X.

Proof

(T1) $X, \phi \in \mathcal{T}_j$ for all j so $X, \phi \in \bigcap_{j \in J} \mathcal{T}_j$.

(T2) Take any finite collection of sets $A_1, ..., A_n \in \bigcap_{j \in J} \mathcal{T}_j$. Then $A_1, ..., A_n \in \mathcal{T}_j$ for each j and so $\bigcap_{i=1}^n A_i \in \mathcal{T}_j$ for each j and so $\bigcap_{i=1}^n A_i \in \bigcap_{j \in J} \mathcal{T}_j$.

(T3) Take any collection of sets $A_i \in \bigcap_{j \in J} \mathcal{T}_j, i \in I$. Then $A_i \in \mathcal{T}_j$, for all $i \in I$ and all $j \in J$ and so $\bigcup_{i \in I} A_i \in \mathcal{T}_j$ for each j and thus $\bigcup_{i \in I} A_i \in \bigcap_{j \in J} \mathcal{T}_j$. ■

Corollary 1.2 *There exists a topology $\mathcal{U}$ on $\mathbb{R}$ containing all intervals (a, b) and such that if $\mathcal{T}_0$ is any other topology containing all such intervals then $\mathcal{U} \subseteq \mathcal{T}_0$.*

Proof There exists at least one topology on $\mathbb{R}$ containing all (a, b), namely $\mathcal{T}_0 = \mathcal{P}(\mathbb{R})$, the power set of $\mathbb{R}$. Set $\mathcal{U} = \bigcap \mathcal{T}$, the intersection over all topologies containing all the intervals (a, b). This is a topology by Lemma 1.1 and the minimality property is immediate from the definition. ■

Definition We say that $\mathcal{U}$ is the *usual topology* on $\mathbb{R}$.

Definition Given any collection $\mathcal{A}$ of subsets of X, the intersection of all topologies containing $\mathcal{A}$ is said to be the topology *generated* by $\mathcal{A}$.

Example 1 The usual topology on $\mathbb{R}$ is the topology *generated* by the intervals (a, b).

There are other topologies on $\mathbb{R}$, such as

Example 2 The *co-finite topology* on $\mathbb{R}$ is defined by $A \in$ *co-finite* if, and only if, either $A = \phi$ or $|A^c| < \infty$. (That is, the complement is finite.)

For the following definition we require the idea of *pre-image* of a map $f : X \to Y$. So if $A \subseteq Y$ then the pre-image of A is given by

$$f^{-1}(A) = \{x \in X : f(x) \in A\}.$$

Definition Given topological spaces $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ and a map $f : X \to Y$ we say that f is *continuous with respect to* $\mathcal{T}_X, \mathcal{T}_Y$ if

$$U \in \mathcal{T}_Y \Rightarrow f^{-1}(U) \in \mathcal{T}_X.$$

(That is, the preimage of an open set is open.)

We will require later the following result.

Theorem 1.3 (Heine-Borel) *If* $[a, b] \subseteq \mathbb{R}$ *is covered by a collection of* (c_i, d_i), *so* $[a, b] \subseteq \bigcup_{i \in I}(c_i, d_i)$, *then there exists a finite sub-collection of the* (c_i, d_i), *which can relabeled as* $1 \le i \le N$ *such that* $[a, b] \subseteq \bigcup_{i=1}^{N}(c_i, d_i)$.

Proof See Appendix. ■

Theorem 1.4 (Lindelöf's Theorem) *If* $\mathcal{G} = \{I_\alpha : \alpha \in A\}$ *is a collection of intervals* $(a, b) \subseteq \mathbb{R}$, *possibly an uncountable collection, then there exists a countable subcollection* $\{I_i : i \ge 1\} \subseteq \mathcal{G}$ *such that*

$$\bigcup_{\alpha \in A} I_\alpha = \bigcup_{i=1}^{\infty} I_i.$$

Proof See Appendix ■

Corollary 1.5

Let $\mathcal{S}$ *be the set of all countable unions of intervals* $(a, b) \subseteq \mathbb{R}$. *Then* $\mathcal{S} = \mathcal{U}$, *the usual topology.*

Proof Let $S \in \mathcal{S}$ so $S = \bigcup_{i=1}^{\infty}(a_i, b_i)$ for some (a_i, b_i) But $(a_i, b_i) \in \mathcal{U}$ for all i, and $\mathcal{U}$ is a topology so $\bigcup_{i=1}^{\infty}(a_i, b_i) \in \mathcal{U}$. Thus $S \in \mathcal{U}$ and so $\mathcal{S} \subseteq \mathcal{U}$.

We now show that $\mathcal{S}$ is a topology by verifying the properties T1, T2 and T3.

(T1) $\mathbb{R} = \bigcup_{n \in \mathbb{N}}(-n, n) \in \mathcal{S}$ and $\phi = (0, 0) \in \mathcal{S}$.

(T2) If $S_1, S_2, S_3, ..., S_n \in \mathcal{S}$ then each

$$S_i = \bigcup_{j_i=1}^{\infty} I_{j_i} \text{ for some } I_{j_i} = (a_{j_i}, b_{j_i}).$$

Then

$$
\begin{aligned}
\bigcap_{i=1}^{n} S_i &= \bigcap_{i=1}^{n} \bigcup_{j_i=1}^{\infty} I_{j_i} \\
&= \left(\bigcup_{j_1=1}^{\infty} I_{j_1} \right) \cap \left(\bigcup_{j_2=1}^{\infty} I_{j_2} \right) \cap ... \cap \left(\bigcup_{j_n=1}^{\infty} I_{j_n} \right) \\
&= \bigcup_{j_1=1}^{\infty} \bigcup_{j_2=1}^{\infty} ... \bigcup_{j_n=1}^{\infty} \left(I_{j_1} \cap I_{j_2} \cap ... \cap I_{j_n} \right).
\end{aligned}
$$

Importantly we have a **finite** intersection of open intervals which is therefore an open interval. (It is possible that an infinite intersection of open intervals is closed.) Thus $\bigcap_{i=1}^{n} S_i$ is a countable union of open intervals, hence it is an element of $\mathcal{S}$.

(T3) If $S_k \in \mathcal{S}$ for $k \in K$, perhaps an uncountable collection, then $\bigcup_{k\in K} S_k = \bigcup_{k\in K} \bigcup_{j_k=1}^{\infty} I_{j_k}$ is a, possibly uncountable, union of intervals $(a_{j_k}, b_{j_{ik}})$. But by Lindelöf's Theorem this can be written as a countable union and so $\bigcup_{k\in K} S_k \in \mathcal{S}$.

Hence $\mathcal{S}$ is **a** topology containing the intervals (a, b). But $\mathcal{U}$ is **the** minimal topology containing these intervals. Hence $\mathcal{U} \subseteq \mathcal{S}$.

Thus $\mathcal{U} = \mathcal{S}$. ■

1.2 Rings

Definition A collection, $\mathcal{S}$, of subsets of the non-empty set X is a *semi-ring* if

(i) $\phi \in \mathcal{S}$,
(ii) $A, B \in \mathcal{S} \Rightarrow A \cap B \in \mathcal{S}$,
(iii) $A, B \in \mathcal{S} \Rightarrow A \setminus B = \bigcup_{i=1}^{N} E_i$, a finite disjoint union of $E_i \in \mathcal{S}$.

Example 3 The collection, $\mathcal{P}$, of all finite intervals of the form $(a, b] \subseteq \mathbb{R}$ form a semi-ring.

This is the most important example of a semi-ring we shall study. It should be compared with the collection of all intervals of the form (a, b) that does **not** form a semi-ring.

Verification is left to the student.

Definition A non-empty collection, $\mathcal{R}$, of subsets of X is a *ring* if

(i) $A, B \in \mathcal{R} \Rightarrow A \cup B \in \mathcal{R}$,
(ii) $A, B \in \mathcal{R} \Rightarrow A \setminus B \in \mathcal{R}$.

Note (i) $\mathcal{R} \neq \phi$ implies that there exists a set A in $\mathcal{R}$ and so $\phi = A \setminus A \in \mathcal{R}$.

*(ii) Such a collection, with the operations

$$A + B = A \bigtriangleup B \quad \text{and} \quad A.B = A \cap B$$

is a ring as defined in the second year course. (Here $A \bigtriangleup B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference.)

Note The collection $\mathcal{P}$ is not a ring, for instance $(0,3]$ and $(1,2] \in \mathcal{P}$ but $(0,3] \setminus (1,2] = (0,1] \cup (2,3] \notin \mathcal{P}$.

Definition The collection, $\mathcal{E}$, of all finite unions of disjoint members of $\mathcal{P}$, is called the set *elementary figures* in $\mathbb{R}$.

Example The collection $\mathcal{E}$ is a ring.

Verification.

Let $A, B \in \mathcal{E}$, so $A = \bigcup_{i=1}^m A_i$ and $B = \bigcup_{j=1}^n B_j$ where $A_i, B_j \in \mathcal{P}$, disjoint unions. Then

$$\begin{aligned} A \setminus B &= A \cap B^c = \left(\bigcup_{i=1}^m A_i \right) \cap B^c \\ &= \bigcup_{i=1}^m (A_i \cap B^c), \end{aligned}$$

a disjoint union. Writing $A_i = (a, b]$ and $B = \bigcup_{j=1}^n (a_j, b_j]$ we see that

$$A_i \cap B^c = (a, b] \cap \left\{ (-\infty, a_1] \cup \bigcup_{j=1}^{n-1} (b_j, a_{j+1}] \cup (b_n, +\infty] \right\}$$

which, on applying the distributive law, is seen to be in $\mathcal{E}$. Then, since the A_i are disjoint we see that $A \setminus B$ is a disjoint union of sets from $\mathcal{E}$ and so $A \setminus B \in \mathcal{E}$, i.e. condition (ii) holds.

Also $A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$, a disjoint union. By what we have already proved it suffices to show that $A \cap B \in \mathcal{E}$. But $A \cap B = \bigcup_{i=1}^m \bigcup_{j=1}^n (A_i \cap B_j)$ a disjoint union of sets from $\mathcal{P}$ and so in $\mathcal{E}$. Thus condition (i) is satisfied. Hence $\mathcal{E}$ is a ring. ■

Definition A ring $\mathcal{R}$ is a σ*-ring* if it is closed under countable unions. That is, given $A_n \in \mathcal{R}, n \geq 1$ then $\bigcup_{n \geq 1} A_n \in \mathcal{R}$.

Note Given $A_n \in \mathcal{R}$ a σ-ring, write $A = \bigcup_{n \geq 1} A_n$. It is trivial that $\bigcap_{n \geq 1} A_n \subseteq A$ but this observation means that we can write

$$\bigcap_{n \geq 1} A_n = A \setminus \left(A \setminus \bigcap_{n \geq 1} A_n \right).$$

Here

$$\begin{aligned} A \setminus \bigcap_{n\geq 1} A_n &= A \cap \left(\bigcap_{n\geq 1} A_n\right)^c = A \cap \left(\bigcup_{n\geq 1} A_n^c\right) \\ &= \bigcup_{n\geq 1}(A \cap A_n^c) = \bigcup_{n\geq 1}(A \setminus A_n). \end{aligned}$$

Since A and $A_n \in \mathcal{R}$ for all $n \geq 1$ we have $A \setminus A_n \in \mathcal{R}$, and so $\bigcup_{n\geq 1}(A \setminus A_n) \in \mathcal{R}$ and thus $A \setminus \left(A \setminus \bigcap_{n\geq 1} A_n\right)$ or $\bigcap_{n\geq 1} A_n \in \mathcal{R}$. Hence σ-rings are closed under countable intersections.

1.3 Fields

Definition A non-empty collection $\mathcal{F}$ of subsets of a non-empty set X is a *field* (or *algebra*) if

(i) $X \in \mathcal{F}$,
(ii) $\mathcal{F}$ is a ring.

Further X is a σ-*field* (or σ-*algebra*) if

(i) $X \in \mathcal{F}$,
(ii) $\mathcal{F}$ is a σ-ring.

Note A field can be defined to satisfy

(i) $X \in \mathcal{F}$,
(ii) If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$,
(iii) If $A, B \in \mathcal{F}$ then $A \cup B \in \mathcal{F}$.

Similarly for σ-fields.

***Verification** that (i) $X \in \mathcal{F}$, (ii) $A, B \in \mathcal{R} \Rightarrow A \cup B \in \mathcal{R}$, and (iii) $A, B \in \mathcal{R} \Rightarrow A \setminus B \in \mathcal{R}$ are equivalent to (i') $X \in \mathcal{F}$, (ii') $A, B \in \mathcal{R} \Rightarrow A \cup B \in \mathcal{R}$,and (iii') if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$.

Assume (i), (ii) and (iii). Then (i') and (ii') obviously holds while (iii') follows from (iii) and (i) with $X, A \in \mathcal{F}$ implying $X \setminus A \in \mathcal{F}$, i.e. $A^c \in \mathcal{F}$.

Assume (i'), (ii') and (iii'). Then (i) and (ii) obviously holds while (iii) follows from $A \setminus B = A \cap B^c = (A^c \cup B)^c$ which is in $\mathcal{R}$ by (ii') and (iii'). ■

Example 5 (a) Let X be an infinite set. Define $\mathcal{F}$ by $A \in \mathcal{F}$ if, and only if, either $|A| < \infty$ or $|A^c| < \infty$. Then $\mathcal{F}$ is a field but it is not a σ-field.

(b) Let X be an infinite set. Define $\mathcal{F}$ by $A \in \mathcal{F}$ if, and only if, either $|A|$ is countable or $|A^c|$ is countable. Then $\mathcal{F}$ is a σ-field.

Theorem 1.5 *The intersection of any non-empty collection of rings, fields or σ-fields in X is, respectively, a ring, field or σ-field.*

Proof is identical in method to that of Lemma 1.1. ■

Corollary 1.6 *Given a collection $\mathcal{A}$ of subsets of X there exists*

(i) *a minimal ring containing $\mathcal{A}$, denoted by $\mathcal{R}(\mathcal{A})$,*
(ii) *a minimal σ-field containing $\mathcal{A}$, denoted by $\sigma(\mathcal{A})$.*

These are minimal in that if

(i) *$\mathcal{R}$ is any ring containing $\mathcal{A}$ then $\mathcal{R}(\mathcal{A}) \subseteq \mathcal{R}$,*
(ii) *$\mathcal{F}$ is a σ-field containing $\mathcal{A}$ then $\sigma(\mathcal{A}) \subseteq \mathcal{F}$.*

Proof Simply choose $\mathcal{R}(\mathcal{A})$ to be the intersection of all rings containing $\mathcal{A}$ and $\sigma(\mathcal{A})$ to be the intersection of all σ-fields containing $\mathcal{A}$. These are non-empty intersections since there is always at least one ring or σ-field on a non-empty set X, namely the power set $P(X)$. ■

Definition We say that $\mathcal{R}(\mathcal{A})$ is the ring *generated* by $\mathcal{A}$. Similarly we say that $\sigma(\mathcal{A})$ is the σ-field *generated* by $\mathcal{A}$, often denoted by $\mathcal{B}(\mathcal{A})$, the *Borel field generated by $\mathcal{A}$.*

Theorem 1.7 *Let $\mathcal{R}(\mathcal{C})$ be the ring generated by the semi-ring $\mathcal{C}$ in X. Then $\mathcal{R}(\mathcal{C})$ is the collection of finite unions of disjoint sets from $\mathcal{C}$, that is*

$$\mathcal{R}(\mathcal{C}) = \left\{ A \subseteq X : A = \bigcup_{i=1}^{n} E_i \textit{ for some disjoint members of } \mathcal{C} \right\}. \qquad (*)$$

Proof Let $\mathcal{A}$ denote the right hand side of (*). If $A \in \mathcal{A}$ then $A = \bigcup_{i=1}^{n} E_i$ for some disjoint members of $\mathcal{C}$. But the ring $\mathcal{R}(\mathcal{C})$ is closed under finite unions and in particular, closed under finite unions of elements of C and so $A \in \mathcal{R}(\mathcal{C})$. Hence $\mathcal{A} \subseteq \mathcal{R}(\mathcal{C})$.

Next we will show that $\mathcal{A}$ is a ring which we do by verifying the definition. Take any $A, B \in \mathcal{R}(\mathcal{C})$ so

$$A = \bigcup_{i=1}^{m} E_i \text{ and } B = \bigcup_{j=1}^{n} F_j$$

for some finite disjoint collections $\{E_i\}$ and $\{F_j\}$ in $\mathcal{C}$. Then

$$
\begin{aligned}
A \setminus B &= \left(\bigcup_{i=1}^{m} E_i\right) \cap \left(\bigcup_{j=1}^{n} F_j\right)^c \\
&= \left(\bigcup_{i=1}^{m} E_i\right) \cap \left(\bigcap_{j=1}^{n} F_j^c\right) \\
&= \bigcup_{i=1}^{m} \left(E_i \cap \left(\bigcap_{j=1}^{n} F_j^c\right)\right) \\
&= \bigcup_{i=1}^{m} \left\{\bigcap_{j=1}^{n} (E_i \setminus F_j)\right\}. \qquad \text{(a)}
\end{aligned}
$$

Yet $\mathcal{C}$ is a semi-ring so

$$
E_i \setminus F_j = \bigcup_{\ell=1}^{L_{ij}} H_{ij\ell},
$$

a disjoint union of sets from $\mathcal{C}$. So

$$
\begin{aligned}
\bigcap_{j=1}^{n} (E_i \setminus F_j) &= \bigcap_{j=1}^{n} \bigcup_{\ell=1}^{L_{ij}} H_{ij\ell} \\
&= \left(\bigcup_{\ell_1=1}^{L_{i1}} H_{i1\ell_1}\right) \cap \left(\bigcup_{\ell_2=1}^{L_{i2}} H_{i2\ell_2}\right) \cap \ldots \cap \left(\bigcup_{\ell_n=1}^{L_{in}} H_{in\ell_n}\right) \\
&= \bigcup_{\ell_1=1}^{L_{i1}} \bigcup_{\ell_2=1}^{L_{i2}} \ldots \bigcup_{\ell_n=1}^{L_{in}} \left(H_{i1\ell_1} \cap H_{i2\ell_2} \cap \ldots \cap H_{in\ell_n}\right), \qquad \text{(b)}
\end{aligned}
$$

a disjoint union. Again since $\mathcal{C}$ is a semi-ring we have

$$
H_{i1\ell_1} \cap H_{i2\ell_2} \cap \ldots \cap H_{in\ell_n} \in \mathcal{C}.
$$

So combining (a) and (b) we see that $A \setminus B$ is a disjoint union of sets from $\mathcal{C}$, that is, $A \setminus B \in \mathcal{A}$

Similarly $E_i \cap F_j \in \mathcal{C}$ and so

$$
\begin{aligned}
A \cap B &= \left(\bigcup_{i=1}^{m} E_i\right) \cap \left(\bigcup_{j=1}^{n} F_j\right) \\
&= \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} (E_i \cap F_j) \\
&\in \mathcal{A}.
\end{aligned}
$$

Then $A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$ is a disjoint union of sets that we have seen above are in $\mathcal{A}$ and so $A \cup B \in \mathcal{A}$. Thus $\mathcal{A}$ is a ring.

Trivially $\mathcal{C} \subseteq \mathcal{A}$. But by definition $\mathcal{R}(\mathcal{C})$ is the smallest ring containing $\mathcal{C}$. Hence $\mathcal{R}(\mathcal{C}) \subseteq \mathcal{A}$.

Combining we get our result, $\mathcal{R}(\mathcal{C}) = \mathcal{A}$. ■

Corollary 1.8 $\mathcal{E} = \mathcal{R}(\mathcal{P})$.

Proof This follows immediately from Theorem 1.7 and the definition of $\mathcal{E}$ as the collection of finite disjoint unions of sets from $\mathcal{P}$, a semi-ring. ■

Definition The *Borel sets* in $\mathbb{R}$ are the elements of the σ-field generated by $\mathcal{P}$. It can be denoted by $\sigma(\mathcal{P}), \mathcal{B}(\mathcal{P})$ or just $\mathcal{B}$.

Theorem 1.9 *In* $\mathbb{R}$ *we have*

(i) $\mathcal{U} \subseteq \mathcal{B}(\mathcal{P})$,
(ii) $\mathcal{P} \subseteq \mathcal{B}(\mathcal{U})$,
(iii) $\mathcal{B}(\mathcal{P}) = \mathcal{B}(\mathcal{U})$.

Proof

(i) Given any $A \in \mathcal{U}$ we can write

$$
\begin{aligned}
A &= \bigcup_{i=1}^{\infty} (a_i, b_i) \qquad \text{for some } a_i \text{ and } b_i, \text{by Corollary 1.5} \\
&= \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} \left(a_i, b_i - \frac{1}{n}\right].
\end{aligned}
$$

Each $(a_i, b_i - 1/n] \in \mathcal{P}$ and so A is contained in any σ-field containing $\mathcal{P}$, in particular $\mathcal{B}(\mathcal{P})$. Hence $\mathcal{U} \subseteq \mathcal{B}(\mathcal{P})$.

(ii) Given any $A \in \mathcal{P}$ we can write

$$
\begin{aligned}
A &= (a, b] \qquad \text{for some } a \text{ and } b, \\
&= \bigcap_{n=1}^{\infty} \left(a, b + \frac{1}{n}\right).
\end{aligned}
$$

Each $(a, b+1/n) \in \mathcal{U}$ and, since σ-fields are closed under countable intersections, A lies in any σ-field containing $\mathcal{U}$, in particular, $\mathcal{B}(\mathcal{U})$. Hence $\mathcal{P} \subseteq \mathcal{B}(\mathcal{U})$.

(iii) Part (i) tells us that $\mathcal{B}(\mathcal{P})$ is a σ-field containing $\mathcal{U}$. But $\mathcal{B}(\mathcal{U})$ is **the** minimal σ-field containing $\mathcal{U}$. Hence $\mathcal{B}(\mathcal{U}) \subseteq \mathcal{B}(\mathcal{P})$. Similarly part (ii) implies $\mathcal{B}(\mathcal{P}) \subseteq \mathcal{B}(\mathcal{U})$. Hence $\mathcal{B}(\mathcal{P}) = \mathcal{B}(\mathcal{U})$. ■

* **Note** For a collection of subsets $\mathcal{A}$ of a set X the σ-field generated by $\mathcal{A}$ need not contain the topology generated by $\mathcal{A}$. This is because topologies contain possibly uncountable unions. Exceptionally, the usual topology, $\mathcal{U}$, in $\mathbb{R}$ is made up from countable unions of intervals, (a, b), by Corollary 1.5. Hence any σ-field containing all intervals (a, b) must also contain $\mathcal{U}$. The proof of Corollary 1.5 depends essentially on Lindelöf's Lemma, Theorem 1.4, and if we look at the proof of Theorem 1.4 we see that the result depends on the existence of a countably dense subset of $\mathbb{R}$, namely $\mathbb{Q}$.