Appendix to Notes 9

Complete Product Spaces

An important application of Fubini's Theorem is in the case $(X, \mathcal{F}, \mu) = (Y, \mathcal{G}, \nu) = (\mathbb{R}, \mathcal{L}, \nu)$, when we can define a product measure on $\mathcal{L} * \mathcal{L}$ and express integrals with respect to the product measure in term of iterated integrals.

Yet there is an alternative way of constructing a measure on $\mathbb{R}^2$**:**

We can look at $\mathcal{P}^2 = \{(a, b] \times (c, d]\}$ with the set function μ_2 giving the area of these figures.

We can show that μ_2 is σ-additive on $\mathcal{P}^2$ (c.f. Theorem 2.1).

Then μ_2 can be extended to $\mathcal{E}^2 = \mathcal{R}(\mathcal{P}^2)$ (c.f. Corollary 2.3).

The outer measure μ_2^* can be defined on all subsets of $\mathbb{R}^2$ (c.f. Theorem 2.4).

The set of μ_2^*-measurable sets $\mathcal{L}^2$ is defined and on which the restriction of μ_2^*, denoted by μ_2 again, is a measure (c.f. Example 8 in section 2.5).

In this way we get a measure space $(\mathbb{R}^2, \mathcal{L}^2, \mu_2)$.

The two measure spaces $(\mathbb{R} \times \mathbb{R}, \mathcal{L} * \mathcal{L})$ and $(\mathbb{R}^2, \mathcal{L}^2)$ are different as can be seen from

Example 1 $\mathcal{L} * \mathcal{L} \neq \mathcal{L}^2$.

Verification Let $T \subseteq [0, 1]$ be a non-measurable set, so $T \notin \mathcal{L}$. Consider $T \times \{0\} \subseteq \mathbb{R}^2$.

Note that $T \times \{0\} \notin \mathcal{L} * \mathcal{L}$ for if it were then, by Theorem 5.2, $(T \times \{0\})^y \in \mathcal{L}$ for all y, and in particular when $y = 0$, we get $T \in \mathcal{L}$, a contradiction.

But $[0, 1] \times \{0\} \in \mathcal{L}^2$, $\mu_2([0, 1] \times \{0\}) = 0$ and $\mathcal{L}^2$ is complete, being derived from an outer measure. So all subsets of $[0, 1] \times \{0\}$ are in $\mathcal{L}^2$, that is, $T \times \{0\} \in \mathcal{L}^2$. ■

This example shows, in particular, that $\mathcal{L} * \mathcal{L}$ is not complete.

Claim $\mathcal{B}(\mathcal{P}^2) \subseteq \mathcal{L} * \mathcal{L} \subseteq \mathcal{L}^2$

Proof of claim If $C \in \mathcal{P}^2$, so $C = (a, b] \times (c, d]$, then $C \in \mathcal{L} \times \mathcal{L}$, i.e. $\mathcal{P}^2 \subseteq \mathcal{L} \times \mathcal{L}$. Thus the smallest σ-field containing $\mathcal{P}^2$ is contained in the smallest containing $\mathcal{L} \times \mathcal{L}$, that is, $\mathcal{B}(\mathcal{P}^2) \subseteq \mathcal{L} * \mathcal{L}$.

Given $D \times E \in \mathcal{L} \times \mathcal{L}$ write $D \times E = (D \times \mathbb{R}) \cap (\mathbb{R} \times E)$. From Theorem 2 of Appendix 3 we have that $D \in \mathcal{L}$ means there exist $P, Q \in \mathcal{B}$, the Borel sets, with $P \subseteq D \subseteq Q$ and $\mu(Q \setminus P) = 0$. But then $P \times \mathbb{R}, Q \times \mathbb{R} \in \mathcal{B}^2$ with $P \times \mathbb{R} \subseteq D \times \mathbb{R} \subseteq Q \times \mathbb{R}$ and $\mu_2(Q \times \mathbb{R} \setminus P \times \mathbb{R}) = 0$. So by Theorem 2 again, we deduce that $D \times \mathbb{R} \in \mathcal{L}^2$. Similarly for $\mathbb{R} \times E$ and so $D \times E = (D \times \mathbb{R}) \cap (\mathbb{R} \times E)\mathcal{L}^2$. Thus $\mathcal{L} \times \mathcal{L} \subseteq \mathcal{L}^2$. In words $\mathcal{L}^2$ is **a** σ-field containing $\mathcal{L} \times \mathcal{L}$ while $\mathcal{L} * \mathcal{L}$ is **the** minimal σ-field containing $\mathcal{L} \times \mathcal{L}$. Hence $\mathcal{L} * \mathcal{L} \subseteq \mathcal{L}^2$. ■

By Theorem 2 of Appendix 3 we have $\mathcal{L}^2 = \overline{\mathcal{B}(\mathcal{P}^2)}$ (see also Example A of that appendix). Hence since, by the claim, $\mathcal{L} * \mathcal{L}$ is stuck in the middle, we must have that $\mathcal{L}^2$ is the completion of $\mathcal{L} * \mathcal{L}$. So we can write $\mathcal{L}^2 = \overline{\mathcal{L} * \mathcal{L}}$ and $\mu_2 = \overline{\mu \times \mu}$.

These observation can be made in general. Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ be complete measure spaces. Choose $A \in \mathcal{F}$, $A \neq \phi$ with $\mu(A) = 0$. Choose $B \subseteq Y$ with $B \notin \mathcal{G}$. Then $A \times B \subseteq A \times Y$, $\mu * \nu(A \times Y) = 0$ but, from Theorem 5.2, $A \times B \notin \mathcal{F} * \mathcal{G}$ Hence, $\mathcal{F} * \mathcal{G}$ is not complete. We can complete the space $(X \times Y, \mathcal{F} * \mathcal{G}, \mu * \nu)$ in the manner described in Appendix 3, to get $(X \times Y, \overline{\mathcal{F} * \mathcal{G}}, \mu * \nu)$, using the same notation for the measure on the completed σ-field.

We would like an extension for Fubini's Theorem that deals with functions $g : X \times Y \to \mathbb{R}^*$ that are $\overline{\mathcal{F} * \mathcal{G}}$-measurable (this increases the collection of applicable functions). Recall from Lemma 3 of section 3 that $g = f + h$ where f is $\mathcal{F} * \mathcal{G}$-measurable and $h = 0$ a.e.(λ) (where $\lambda = \mu * \nu$). So $\int g d\lambda = \int f d\lambda$ and we can apply Fubini's theorem to this integral over f. We want to write our results in terms of g so we need to know something about h_x and h^y.

Lemma *Let $h : X \times Y \to \mathbb{R}^*$ be an $\overline{\mathcal{F} * \mathcal{G}}$-measurable function such that* $h = 0$ a.e.$(\mu * \nu)$.

Then for almost all x we have $h(x, y) = 0$ for almost all y. In particular h_x is $\mathcal{G}$-measurable.

Similarly for h^y.

Proof Let

$$P = \{(x, y) \in X \times Y : h(x, y) \neq 0\}.$$

So $P \in \overline{\mathcal{F} * \mathcal{G}}$ and $\mu * \nu(P) = 0$. By completion there exists $Q \in \mathcal{F} * \mathcal{G}$ such that $P \subseteq Q$ and $\mu * \nu(Q) = 0$. By the definition of $\mu * \nu$ this means that

$$\int_X \nu(Q_x) d\mu = 0.$$

Since $\nu(Q_x) \geq 0$ we have that $\nu(Q_x) = 0$ a.e.(μ). Thus we have $P_x \subseteq Q_x$, subsets in the complete space $(Y, \mathcal{G}, \nu)$. Hence $P_x \in \mathcal{G}$ a.e.(μ) and $\nu(P_x) = 0$ a.e.(μ), which is, $\nu(\{y \in Y : h_x(y) \neq 0\}) = 0$. ■

Returning to our extension of Fubini's Theorem we apply the known result to f and then rewrite the resulting iterated integrals over g_x and g^y since the lemma says that the iterated integrals over h_x and h^y are zero. Thus

Theorem *Let* $(X, \mathcal{F}, \mu)$ *and* $(Y, \mathcal{G}, \nu)$ *be complete* σ*-finite measure spaces. Let* $g : X \times Y \to \mathbb{R}^*$ *be an* $\overline{\mathcal{F} * \mathcal{G}}$*-measurable function. Then all conclusions of Theorem 5.4 hold with the following differences. That* g_x *is only* $\mathcal{G}$*-measurable* a.e.(μ)*, so* $\alpha(x)$ *is only defined a.e.*(μ)*.*