# Appendix to Notes 5

## An interesting Simple Function

Let $x \in [0,1]$ have the expansion $x = 0.x_1x_2x_3x_4....$ in base $\ell$ for some integer $\ell$. In the cases of ambiguity choose the non-terminating expansion. Here $x_i \in \{0,1,2,...,\ell\}$ for all $i \geq 1$.

For each $i \geq 1$ define $a_i(x) = x_i$. This is obviously a simple function.

**Example** In the special case $\ell = 2$ we have

$$a_1(x) = \begin{cases} 0 & \text{on } (0,1/2], \\ 1 & \text{on } (1/2,1], \end{cases}$$

on noting that because of our convention we write

$$\frac{1}{2} = \frac{0}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + ... \, .$$

Similarly

$$a_2(x) = \begin{cases} 0 & \text{on } (0,1/4] \cup (1/2,3/4], \\ 1 & \text{on } (1/4,1/2] \cup (3/4,1], \end{cases}$$

and

$$a_3(x) = \begin{cases} 0 & \text{on } (0,1/8] \cup (1/4,3/8] \cup (1/2,5/8] \cup (3/4,7/8], \\ 1 & \text{on } (1/8,1/4] \cup (3/8,1/2] \cup (5/8,3/4] \cup (7/8,1]. \end{cases}$$

In general, when $\ell = 2$,

$$a_i(x) = 1 \text{ on } \bigcup_{i=0}^{2^{i-1}-1} \left( \frac{2\ell+1}{2^i}, \frac{2\ell+2}{2^i} \right], \text{ and 0 elsewhere.}$$

To see if $a_i$ is measurable for any $\ell$ take any $0 \leq \kappa \leq \ell - 1$. Then, those $x \in [0,1]$ with $a_i(x) = \kappa$ must be of the form

$$\begin{aligned} x &= \frac{a_1}{\ell} + \frac{a_2}{\ell^2} + ... + \frac{a_{i-1}}{\ell^{i-1}} + \frac{\kappa}{\ell^i} + y, \\ &= \frac{a_1\ell^{i-2} + a_2\ell^{i-3} + ... + a_{i-1}}{\ell^{i-1}} + \frac{\kappa}{\ell^i} + y \end{aligned}$$

where $a_j \in \{0,1,...,\ell-1\}$ for $1 \leq j \leq i-1$, and $0 < y \leq 1/\ell^i$. The terms $a_1\ell^{i-2} + a_2\ell^{i-3} + ... + a_{i-1}$ are distinct and run through all integers from 0 to $(\ell-1)\sum_{j=0}^{i-2} \ell^j = \ell^{i-1} - 1$. Thus

$$
\begin{aligned}
\{x \ \in \ & [0,1] : a_i(x) = \kappa\} \\
= & \bigcup_{n=0}^{\ell^{i-1}-1} \left( \frac{n\ell + \kappa}{\ell^i}, \frac{n\ell + \kappa + 1}{\ell^i} \right],
\end{aligned}
$$

trivially measurable.

**Example**

Let $K$ be the Cantor set. Define $f : [0,1] \to K$ by

$$\sum_{i=1}^{\infty} \frac{x_i}{2^i} \mapsto \sum_{i=1}^{\infty} \frac{2x_i}{3^i}.$$

In the notation above, with $\ell = 2$, we have

$$f(x) = \sum_{i=1}^{\infty} \frac{2a_i(x)}{3^i} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2a_i(x)}{3^i},$$

that is, the limit of an increasing sequence (since $a_i(x) \geq 0$) of Lebesgue measurable function. Hence $f$ is Lebesgue measurable. The function $f$ is called *Cantor's function.*

Interestingly this function maps $[0,1]$, a set of non-zero measure, into $K$ a set of measure zero. This gives the possibility of "problems". This is exploited in the next result. It should be noted that $f$ is one-to-one.

**Theorem**

*The collection of Borel sets is a proper subset of the collection Lebesgue measurable sets.*

**Proof** Assume that every Lebesgue measurable set is a Borel set.

Let $V \subseteq [0,1]$ be a non-Lebesgue measurable set. Then $f(V) \subseteq K$ is a subset of a measurable set of measure zero hence, since Lebesgue measure is complete, $f(V)$ is measurable. By assumption, therefore, $f(V)$ is a Borel set. But then, since $f$ is a measurable function we have that $f^{-1}(f(V))$ is a measurable set. Yet $f$ is one-to-one and so $f^{-1}(f(V)) = V$. So we deduce that $V$ is measurable. This is a contradiction so our assumption is false. Hence there exist Lebesgue measurable sets that are not Borel sets. ■